16.4 Green’s Theorem

Theorem 1

\( \int_C \mathbf{F} \cdot dr \) is independent of path in \( D \) if and only if

\[ \int_C \mathbf{F} \cdot dr = 0 \]

for every closed path \( C \) in \( D \).

In the last section we found out that if \( \mathbf{F} \) was a conservative vector field then we had a nice way to integrate it over a curve. As long as the initial and terminal points were the same, the integral did not depend on the path chosen. But what happens if \( \mathbf{F} \) is not conservative? In 16.2 we needed to evaluate the integral the hard way

\[ \int_C \mathbf{F} \cdot dr = \int_a^b \mathbf{F}(r(t)) \cdot r'(t) \, dt \]

Fortunately, in this section we can evaluate the integral \( \int_C \mathbf{F} \cdot dr \) easily even if \( \mathbf{F} \) is not conservative. But there are conditions on the domain and path.

Theorem 2: Green’s Theorem

Let \( C \) be a positively oriented, piecewise-smooth, simple closed curve in the plane and let \( D \) be the region bounded by \( C \). If \( P \) and \( Q \) have continuous partial derivatives on an open region that contains \( D \), then

\[ \int_C P(x, y) \, dx + Q(x, y) \, dy = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]

The line integral can be converted into a double integral from chapter 15.
Example 1

Evaluate \( \int_C x^4 \, dx + xy \, dy \) where \( C \) is the triangle formed by the points \((0, 0), (1, 0)\) and \((1, 0)\).

1. Let’s look at the path \( C \) and the region \( D \) formed by the triangle.

![Diagram of the triangle](image)

2. Let’s solve this using the direct method from 16.2 by integrating over the curves \( C_1 \), \( C_2 \), and \( C_3 \) separately.

(a) Over \( C_1 \)

i. \( r(t) = (x(t), y(t)) = (0, 1 - t) \), \( 0 \leq x \leq 1 \)

ii. \( r'(t) = (x'(t), y'(t)) = (0, -1) \)

iii. Formulas Needed:

\[
\int_C P(x, y) \, dx = \int_a^b P(x(t), y(t))x'(t) \, dt
\]

\[
\int_C Q(x, y) \, dy = \int_a^b Q(x(t), y(t))y'(t) \, dt
\]

iv. Formula: \( \int_{C_1} \mathbf{F} \cdot dr = \int_a^b \mathbf{F}(r(t)) \cdot r'(t) \, dt = \int_a^b P(x, y) \, dx + Q(x, y) \, dy \)
\[
\int_{C_1} x^4 \, dx + xy \, dy = \int_0^1 (0)^4(0 \, dt) + 0(1 - t)(-1 \, dt) \\
= \int_0^1 0 \, dt \\
= 0
\]

(b) Over \(C_2\)

i. \(r(t) = \langle x(t), y(t) \rangle = \langle t, 0 \rangle, \ 0 \leq t \leq 1\)

ii. \(r'(t) = \langle x'(t), y'(t) \rangle = \langle 1, 0 \rangle\)

\[
\int_{C_2} x^4 \, dx + xy \, dy = \int_0^1 t^4(1 \, dt) + t(0)(0 \, dt) \\
= \int_0^1 t^4 \, dt \\
= \frac{1}{5} t^5 \bigg|_0^1 \\
= \frac{1}{5}
\]

(c) Over \(C_3\)

i. \(r(t) = \langle x(t), y(t) \rangle = \langle 1 - t, t \rangle\)

ii. \(r'(t) = \langle x'(t), y'(t) \rangle = \langle -1, 1 \rangle\)

\[
\int_{C_3} x^4 \, dx + xy \, dy = \int_0^1 (1 - t)^4(-1 \, dt) + (1 - t)t(1 \, dt) \\
= \int_0^1 -(1 - t)^4 + t - t^2 \, dt \\
= \frac{1}{5} (1 - t)^5 + \frac{1}{2} t^2 - \frac{1}{3} t^3 \bigg|_0^1 \\
= -\frac{1}{30}
\]

(d) \(\int_{C_1+C_2+C_3} x^4 \, dx + xy \, dy = 0 + \frac{1}{5} - \frac{1}{30} = \frac{1}{6}\)
3. Let’s try with Green’s Theorem

(a) Let \( P = x^4 \) and \( \frac{\partial P}{\partial y} = 0 \)

(b) Let \( Q = xy \) and \( \frac{\partial Q}{\partial x} = y \)

(c) \( D = \{(x, y) \mid 0 \leq x \leq 1, \ 0 \leq y \leq 1 - x\} \)

\[
\int_C x^4 \, dx + xy \, dy = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]
\[
= \int \int_D y \, dA
\]
\[
= \int_0^1 \int_0^{1-x} y \, dy \, dx
\]
\[
= \int_0^1 \left[ \frac{1}{2} y^2 \right]^{1-x}_0 \, dx
\]
\[
= \int_0^1 \frac{1}{2} (1 - x)^2 \, dx
\]
\[
= \frac{1}{6} (1 - x)^3 \bigg|_0^1
\]
\[
= \frac{1}{6}
\]

Example 2

Use Green’s Theorem to evaluate \( \int_C (3y - e^{\sin x}) \, dx + (7x + \sqrt{y^4 + 1}) \, dy \), where \( C \) is the circle \( x^2 + y^2 = 9.1 \)

1. Let \( P = 3y - e^{\sin x} \) and \( \frac{\partial P}{\partial y} = 3 \)

2. Let \( Q = 7x + \sqrt{y^4 + 1} \) and \( \frac{\partial Q}{\partial x} = 7 \)
3. Sketch D

\[ D = \{(r, \theta) \mid 0 \leq r \leq 3, \ 0 \leq \theta \leq 2\pi\} \]

4. The region \( D \) is best described in polar. So we need to change the integral to polar using

\[
x = r \cos \theta
\]
\[
y = r \sin \theta
\]
\[
x^2 + y^2 = r^2
\]

\[
\int_C (3y - e^{\sin x}) \, dx + (7x + \sqrt{y^4 + 1}) \, dy = \int \int_D (7 - 3) \, dA
\]
\[
= \int \int_D 4 \, dA
\]
\[
= \int_0^{2\pi} \int_0^3 4r \, dr \, d\theta
\]
\[
= \int_0^{2\pi} \left[ 2r^2 \right]_0^3 \, d\theta
\]
\[
= \int_0^{2\pi} 18 \, d\theta
\]
\[
= 18\theta \bigg|_0^{2\pi}
\]
\[
= 36\pi
\]
Example 3

Use Green’s Theorem to evaluate \[ \int_C y^2 \, dx + 3xy \, dy \] where \( C \) is the boundary of the semicircular region \( D \) is the upper half plane between \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).

1. \( P = y^2 \) and \( \frac{\partial P}{\partial y} = 2y \)

2. \( Q = 3xy \) and \( \frac{\partial Q}{\partial x} = 3y \)

3. Sketch \( D \)

\[
D = \{ (r, \theta) \mid 1 \leq r \leq 2, \ 0 \leq \theta \leq \pi \}
\]

The region is best described in polar.

\[
\int_C y^2 \, dx + 3xy \, dy = \int \int_D (3y - 2y) \, dA \\
= \int \int_D y \, dA \\
= \int_0^\pi \int_1^2 r \sin(\theta) \cdot r \, dr \, d\theta \\
= \int_0^\pi \sin \theta \, d\theta \cdot \int_1^2 r^2 \, dr \\
= -\cos \theta \bigg|_0^\pi + \frac{1}{3} r^3 \bigg|_1^2 \\
= 2 \cdot \frac{7}{3} \\
= \frac{14}{3}
\]
Example 4

Use Green's Theorem to evaluate \( \int_C x^2y^2 \, dx + xy \, dy \) where \( C \) is the arc of \( y = x^2 \) from \((0,0)\) to \((1,1)\), line segments from \((1,1)\) to \((0,1)\) and from \((0,1)\) to \((0,0)\).

1. \( P = x^2y^2 \) and \( \frac{\partial P}{\partial y} = 2x^2y \)

2. \( Q = xy \) and \( \frac{\partial Q}{\partial x} = y \)

3. Sketch \( D \)

\[
D = \{(x, y) \mid 0 \leq x \leq 1, \ x^2 \leq y \leq 1\}
\]

\[
\int_C x^2y^2 \, dx + xy \, dy = \int \int_D y - 2x^2y \, dA
\]

\[
= \int_0^1 \int_{x^2}^1 y - 2x^2y \, dy \, dx
\]

\[
= \int_0^1 \left[ \frac{1}{2}y^2 - x^2y \right]_{x^2}^1 \, dx
\]

\[
= \int_0^1 x^6 - \frac{1}{2}x^4 - x^2 + \frac{1}{2} \, dx
\]

\[
= \frac{1}{7}x^7 - \frac{1}{10}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x^2 \Bigg|_0^1
\]

\[
= \frac{22}{105}
\]