15.7 Triple Integrals in Cylindrical Coordinates

In 2D this would be called Polar Coordinates. When extending it to 3D, by adding the $z$-axis, we represent points $(x, y, z)$ as $(r, \theta, z)$.

**Definition 1: Convert Coordinates**

**Cylindrical to Rectangular Coordinates**

\[ x = r \cos(\theta), \quad y = r \sin(\theta), \quad z = z \]

**Rectangular to Cylindrical Coordinates**

\[ r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z \]

**Example 1**

Plot \( \left( 2, \frac{2\pi}{3}, 1 \right) \) and find the rectangular coordinates.
Example 2

Change to Cylindrical Coordinates: \((3, -3, -2)\)

\[
r^2 = (3)^2 + (-3)^2 = 18 \Rightarrow r = \sqrt{18}
\]
\[
\tan(\theta) = \frac{3}{-3} = -1 \Rightarrow \theta = -\pi/4
\]
\[
z = -2
\]
Definition 2: Triple Integrals in Cylindrical Coordinates

\[ E = \begin{cases} 
\alpha \leq \theta \leq \beta \\
 r_1(\theta) \leq r \leq r_2(\theta) \\
 u_1(r \cos \theta, r \sin \theta) \leq z \leq u_2(r \cos \theta, r \sin \theta) 
\end{cases} \]

\[
\int \int \int_E f(x, y, z) \, dV = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta) r \, dz \, dr \, d\theta
\]

Example 3

Find the volume of the potion of the surface that lies below \( z = 1 - x^2 - y^2 \) and above the \( xy \)-plane.

Let’s take a look at the graph and the projection of \( E \) onto the \( xy \) plane.
We are trying to evaluate \( \int \int \int_E 1 \, dV \)

1. Let’s start with trying to write \( E \)

\[
E = \begin{cases} 
0 \leq \theta \leq 2\pi \\
0 \leq r \leq 1 \\
0 \leq z \leq 1 - x^2 - y^2 
\end{cases}
\]

Notice how the inequalities involve \( x \) and \( y \). If we want to convert this triple integral to cylindrical coordinates we need to rewrite \( x \) and \( y \) using the conversion formulas from above.

\[
1 - x^2 - y^2 = 1 - (x^2 + y^2) = 1 - r^2
\]

2. Now express \( E \) as

\[
E = \begin{cases} 
0 \leq \theta \leq 2\pi \\
0 \leq r \leq 1 \\
0 \leq z \leq 1 - r^2 
\end{cases}
\]

3. Set up the integral. Recall that if you’re looking for th volume then \( f(x, y, z) = 1 \)

\[
\int_0^{2\pi} \int_0^1 \int_{1-r^2}^{1-r^2} 1 \, r \, dz \, dr \, d\theta
\]
4. Evaluate the inside integral
\[ \int_0^{1-r^2} rz \, dz = r(1 - r^2) = r - r^3 \]
\[ \int_0^{2\pi} \int_0^1 r - r^3 \, dr \, d\theta \]

5. Evaluate the inside integral
\[ \int_0^1 r - r^3 \, dr = \frac{1}{2}r^2 - \frac{1}{4}r^4 \bigg|_0^1 = \frac{1}{4} \]
\[ \int_0^{2\pi} \frac{1}{4} \, d\theta \]

6. Evaluate the last integral
\[ \int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{1}{4} \theta \bigg|_0^{2\pi} = \frac{\pi}{2} \]

Example 4

Setup \( \int \int \int_E x + y + z \, dV \) where \( E \) is the solid in the first octant that lies under \( z = 4 - x^2 - y^2 \) as a triple integral in cylindrical coordinates.

If you want to project the surface onto the \( xy \) plane, you get

\[ D = \begin{cases} 
0 \leq \theta \leq \pi/2 \\
0 \leq r \leq 2 
\end{cases} \]
1. Now we can express $E$ as

$$E = \begin{cases} 
0 \leq \theta \leq \pi/2 \\
0 \leq r \leq 2 \\
0 \leq z \leq 4 - x^2 - y^2 
\end{cases}$$

2. Convert to Cylindrical Coordinates

$$E = \begin{cases} 
0 \leq \theta \leq \pi/2 \\
0 \leq r \leq 2 \\
0 \leq z \leq 4 - r^2 
\end{cases}$$

3. The integral can be written as

$$\int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r \cos \theta + r \sin \theta + z) r \, dz \, dr \, d\theta$$

$$\int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} r^2 \cos \theta + r^2 \sin \theta + zr \, dz \, dr \, d\theta$$

Example 5

Sketch the region $E$ represented by the integral

$$\int \int \int_E x^2 + y^2 \, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2}} x^2 + y^2 \, dz \, dy \, dx$$

Then Evaluate the integral in cylindrical coordinates.

The projection of $E$ onto the $xy$ plane will be

$$D = \begin{cases} 
-2 \leq x \leq 2 \\
-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2} 
\end{cases}$$
We can rewrite the projection region $D$ in cylindrical coordinates by

$$D = \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2 \end{cases}$$

Adding in $z$ the region $E$ is expressed as

$$E = \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2 \\ \sqrt{x^2 + y^2} \leq z \leq 2 \end{cases}$$

$$\int_0^{2\pi} \int_0^2 \int_r^{\sqrt{x^2 + y^2}} r^2 \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3 \, dz \, dr \, d\theta$$

1. Evaluate inside integral

$$\int_r^2 r^3 \, dz = r^3 \bigg|_r^2 = 2r^3 - r^4$$

$$\int_0^{2\pi} \int_0^2 2r^3 - r^4 \, dr$$

2. Evaluate the inside integral

$$\int_0^2 2r^3 - r^4 \, dr = \frac{1}{2}r^4 - \frac{1}{5}r^5 \bigg|_0^2 = \frac{8}{5}$$

$$\int_0^{2\pi} \frac{8}{5} \, d\theta$$

$$= \frac{8}{5} \bigg|_0^{2\pi} = \frac{16\pi}{5}$$

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