EXAM 3 MATH 230, CALCULUS II NAME: Solution

Show all work to receive full credit.

1. Use the an appropriate convergence / divergence test to determine whether the series converges or diverges. You must name the test you used.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ Let $a_n = (-1)^n \cdot \frac{1}{\ln n}$. and $b_n = \frac{1}{\ln n}$.

(1) b_n is positive ✓

(2) b_n is decreasing ✓ Show $(b_n)' < 0$ for large n.

(3) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0$ ✓

By the Alternating Series Test, $\sum \frac{(-1)^n}{\ln n}$ converges

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-2}{n^2+1}\right)^n$. Try Root Test

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n-2}{n^2+1}\right)^n\right|} = \lim_{n\to\infty} \frac{n-2}{n^2+1} = 0 < 1$$

So $\displaystyle\sum_{n\to\infty} \left(\frac{n-2}{n^2+1}\right)^n$ converges by the Root Test

(c) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{5n}\right)^n$. If you try the root test, $L = 1$ (inconclusive)

Let's Make sure $\displaystyle\lim_{n\to\infty} \left(1+\frac{1}{5n}\right)^n = 0$

• $\displaystyle\lim_{n\to\infty} \left(1+\frac{1}{5n}\right)^n = 1^\infty$ (this is a LH problem): $\left(1+\frac{1}{5n}\right)^n = e^{n \ln\left(1+\frac{1}{5n}\right)}$

• $$\lim_{n\to\infty} n \ln\left(1+\frac{1}{5n}\right) = \lim_{n\to\infty} \frac{\ln(1+1/5n)}{\frac{1}{n}} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{1+1/5n} \cdot \frac{-1}{5n^2}}{\frac{-1}{5n^2}} = \lim \frac{1}{1+\frac{1}{5n}} \cdot \frac{1}{5}$$

$$= \lim_{n\to\infty} \frac{1}{5+\frac{1}{n}} = \frac{1}{5}$$ [redacted] And $e^{1/5} \neq 0$ So $\displaystyle\sum_{n=1} \left(1+\frac{1}{5n}\right)^n$ diverges

by the Divergence Test

2. Determine whether each series converges or diverges by using one or more of our tests. Make your arguments carefully and fully.

(a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Try Integral Test.

• $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$. Let $u = \ln x$, $du = \frac{1}{x}dx$. If $x = \infty$, $u = \infty$
$x = 2$, $u = \ln 2$

• $\int_{\ln 2}^{\infty} \frac{1}{u^2}\,du = \int_{\ln 2}^{\infty} u^{-2}\,du = \left.\frac{u^{-1}}{-1}\right|_{\ln 2}^{\infty} = \left.\frac{-1}{u}\right|_{\ln 2}^{\infty}$

If you do this problem like the textbook, you should have

$$\lim_{b\to\infty}\left[\left.\frac{-1}{u}\right|_{\ln 2}^{b}\right] = \lim_{b\to\infty}\left[\frac{-1}{b} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2}$$

Since $\int_2^{\infty} \frac{1}{x(\ln x)^2}dx$ converges, then so does $\sum_{n=2} \frac{1}{n(\ln n)^2}$. (By Integral Test)

(b) $\sum_{n=2}^{\infty} \frac{n!3^n}{(2n)!}$ Try Ratio Test

• $a_n = \frac{n!\,3^n}{(2n)!}$ $\quad a_{n+1} = \frac{(n+1)!\,3^{n+1}}{(2(n+1))!} = \frac{(n+1)!\,3^{n+1}}{(2n+2)!}$

• $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,3^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n!\,3^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot n!\cdot 3^n\cdot 3}{(2n+2)(2n+1)(2n)!}\cdot\frac{(2n)!}{n!\,3^n}\right|$

$= \lim_{n\to\infty}\left|\frac{(n+1)\cdot 3}{(2n+2)(2n+1)}\right| = \lim_{n\to\infty}\left|\frac{3n+3}{4n^2+6n+2}\right| = 0 \leftarrow L$

Since $L<1$, by the Ratio Test, $\sum_{2}^{\infty} \frac{n!\,3^n}{(2n)!}$ converges

(c) $\sum_{n=1} \frac{\sin(n+2) + \tan^{-1}(n)}{n^2}$ Note: $|\sin(n+2)| < 1$ and $|\tan^{-1} n| < \pi/2$

so $|\sin(n+2) + \tan^{-1} n| < 1 + \pi/2$

(1) $\sum_1^{\infty} \left| \frac{\sin(n+2) + \tan^{-1} n}{n^2} \right| \leq \sum_1^{\infty} \frac{1+\pi/2}{n^2} = (1+\pi/2) \sum \frac{1}{n^2}$

which converges by p-test. So $\sum_1^{\infty} \left| \frac{\sin(n+2) + \tan^{-1} n}{n^2} \right|$

converges by DCT.

(2) Since $\sum \left| \frac{\sin(n+2) + \tan^{-1} n}{n^2} \right|$ converges, so does $\sum_1^{\infty} \frac{\sin(n+2) + \tan^{-1} n}{n^2}$

(3) Note: If $\sum |a_n|$ converges, so does $\sum a_n$

3. Sum the series or show that they diverge. Display your work carefully and completely.

(a) $\sum_{n=1}^{\infty} \frac{9 + (-3)^n}{5^{n-1}} = \sum_1 \frac{9}{5^{n-1}} + \sum_1 \frac{(-3)^n}{5^{n-1}}$

(1) $\sum_1 \frac{9}{5^{n-1}} = \sum_{n=0} \frac{9}{5^n} = \sum_{n=0} 9\left(\frac{1}{5}\right)^n = \frac{9}{1 - 1/5} = \frac{45}{4}$

(2) $\sum_{n=1} \frac{(-3)^n}{5^{n-1}} = \sum_{n=0} \frac{(-3)^{n+1}}{5^n} = \sum_{n=0} \frac{(-3)\cdot(-3)^n}{5^n} = \sum_{n=0} (-3)\left(\frac{-3}{5}\right)^n$

$= \frac{-3}{1 + 3/5} = \frac{-3}{8/5} = \frac{-15}{8}$

Final: $(1) + (2) = \frac{45}{4} - \frac{15}{8} = \frac{75}{8}$

(b) $\sum_{n=2}^{\infty} \frac{1}{(4n+1)(4n+5)}$

Note: you have two options

(1) Do LCT with $\sum \frac{1}{n^2}$

(2) Integral Test with Partial Fractions

(1) Let $b_n = \frac{1}{n^2}$. $\lim_{n\to\infty} \frac{\frac{1}{(4n+1)(4n+5)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(4n+1)(4n+5)} = \frac{1}{16} > 0$

Since $\sum \frac{1}{n^2}$ converges by p-test, $\sum \frac{1}{(4n+1)(4n+5)}$ conv by LCT

(2) $\int_2^{\infty} \frac{1}{(4x+1)(4x+5)} dx \xrightarrow[\text{Fractions}]{\text{Partial}} \int_2^{\infty} \frac{1/4}{4x+1} - \frac{1/4}{4x+5} dx = \frac{1}{16}\ln|4x+1| - \frac{1}{16}\ln|4x+5|$

$= \frac{1}{16}\ln\left|\frac{4x+1}{4x+5}\right|\Big|_2^{\infty} = \frac{1}{16}\ln\frac{4}{4} - \frac{1}{16}\ln\left|\frac{9}{13}\right| = -\frac{1}{16}\ln\left|\frac{9}{13}\right|$

Converges by Integral Test

4. Compute the integral $\int_1^{\infty} \frac{x^2}{x^3+1}\, dx$

(1) Let $u = x^3+1$, $du = 3x^2\,dx \to \frac{1}{3}du = x^2 dx$

(2) If $x \to \infty$, $u \to \infty$
If $x = 1$, $u = 2$

(3) $\int_1^{\infty} \frac{x^2}{x^3+1} dx = \int_2^{\infty} \frac{1}{3}\cdot\frac{1}{u}\, du = \frac{1}{3}\ln|u|\Big|_2^{\infty}$

$= \lim_{t\to\infty} \left[\frac{1}{3}\ln|u|\Big|_2^{t}\right] = \lim_{t\to\infty}\left[\frac{1}{3}\ln|t| - \frac{1}{3}\ln|2|\right]$

$= \infty - \frac{1}{3}\ln 2$

$= \infty$ Diverges

5. Determine whether the series diverges, converges absolutely, or converges conditionally. Support your answer with a clear argument with one or more tests.

(a) $\sum \frac{(-1)^n n^2}{n^{5/2} + \ln n + 2}$ Let $b_n = \frac{n^2}{n^{5/2} + \ln n + 2}$

Since b_n is positive, decreasing, and $\lim b_n = 0$, then $\sum \frac{(-1)^n n^2}{n^{5/2} + \ln n + 2}$ converges. Now we check for absolute convergence
↑ by AST

$\sum \frac{n^2}{n^{5/2} + \ln n + 2}$. Do a LCT with $\sum \underbrace{\frac{1}{n^{1/2}}}_{a_n}$ (which diverges by p-test)

$$\lim_{n\to\infty} \frac{n^2}{n^{5/2} + \ln n + 2} \cdot \frac{n^{1/2}}{1} = \lim_{n\to\infty} \frac{n^{5/2}}{n^{5/2} + \ln n + 2} = \lim_{n\to\infty} \frac{1}{1 + \frac{\ln n}{n^{5/2}} + \frac{2}{n^{5/2}}} = 1$$

Since $\lim_{n\to\infty} \frac{b_n}{a_n} = L = 1$ (a finite #), then $\sum \frac{n^2}{n^{5/2} + \ln n + 2}$ [redacted] diverges

Therefore $\sum \frac{(-1)^n n^2}{n^{5/2} + \ln n + 2}$ converges [redacted] conditionally

(b) $\sum \frac{n^2 + 3}{(-e)^n}$

↑ Try Ratio Test

$a_n = \frac{n^2+3}{(-e)^n}$, $a_{n+1} = \frac{(n+1)^2+3}{(-e)^{n+1}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2+3}{(-e)^{n+1}} \cdot \frac{(-e)^n}{n^2+3} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2+3}{n^2+3} \cdot \frac{(-e)^n}{(-e)^n \cdot (-e)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2+3}{n^2+3} \cdot \frac{1}{-e} \right| = 1 \cdot \frac{1}{e} = \frac{1}{e} < 1 = L$$ Since

$L < 1$, $\sum \frac{n^2+3}{(-e)^n}$ converges absolutely