5.10 Taylor and Maclaurin Series

Consider the following power series representation:

\[ f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \ldots \]

In the last section, we could only write functions into a power series if we could get \( f(x) \) into the form \( \frac{a}{1 - u} \) by differentiating or integrating. A natural question is, ”Is there a formula for \( c_n \) based on \( f(x) \)?” One that works for any function \( f(x) \).

We do this by finding a pattern while taking derivatives of \( f(x) \).

1. \( f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \ldots \)
   \[ f(a) = c_0 \]
2. \( f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \ldots \)
   \[ f'(a) = c_1 \]
3. \( f''(x) = 2c_2 + 3 \cdot 2c_3(x - a) + 4 \cdot 3c_4(x - a)^2 + \ldots \)
   \[ f''(a) = 2c_2 \]
   \[ c_2 = \frac{f''(a)}{2} \]
4. \( f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x - a) + \ldots \)
   \[ f'''(a) = 3 \cdot 2c_3 \]
   \[ c_3 = \frac{f'''(a)}{3 \cdot 2} \]
5. \( f^4(x) = 4 \cdot 3 \cdot 2c_4 + \ldots \)
   \[ f^4(a) = 4 \cdot 3 \cdot 2c_4 \]
\[ c_4 = \frac{f^4(a)}{4 \cdot 3 \cdot 2} \]

It appears that if \( f(x) \) has a power series representation, then

\[ c_n = \frac{f^n(a)}{n!} \]

The next theorem will pretty much state the same thing, but a bit more formally.

**Theorem 5.8.** If \( f \) has a power series expansion at \( x = a \), that is, if

\[ f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \]

with \( |x-a| < R \), then

\[ c_n = \frac{f^n(a)}{n!} \]

Plug everything back in and we get:

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n \]

\[ = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \ldots \]

This is called the **Taylor Series** of \( f \) centered at \( x = a \).

When \( a = 0 \), we get the **Maclaurin Series**.

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]
Let’s see how this works. Let’s find the Maclaurin Series of \( f(x) = \frac{1}{1-x} \). Recall we that 
\[
\frac{1}{1-x} = \sum_{0}^{\infty} x^n.
\]

1. Set up a table that organizes the derivatives and \( c_n \)s.

\[
\begin{align*}
  f(x) &= \frac{1}{1-x} \quad \rightarrow \quad c_0 = \frac{f(0)}{0!} = 1 \\
  f'(x) &= \frac{1}{(1-x)^2} \quad \rightarrow \quad c_1 = \frac{f'(0)}{1!} = 1 \\
  f''(x) &= \frac{2}{(1-x)^3} \quad \rightarrow \quad c_2 = \frac{f''(0)}{2!} = 1 \\
  f'''(x) &= \frac{6}{(1-x)^4} \quad \rightarrow \quad c_3 = \frac{f'''(0)}{3!} = 1 \\
  f^4(x) &= \frac{24}{(1-x)^5} \quad \rightarrow \quad c_4 = \frac{f^4(0)}{4!} = 1 \\
  &\vdots
\end{align*}
\]

2. Next, we try to come up with a formula for \( c_n \). I think it’s safe to say

\[ c_n = 1 \]

3. Write out the terms of the series

\[
 f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots
\]

\[
 f(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots
\]

4. Put it back into series notation

\[
 f(x) = \sum_{0}^{\infty} x^n
\]
Example 5.67. Find the Maclaurin Series of \( f(x) = e^x \) and its radius of convergence.

1. Set up a table that organizes the derivatives and \( c_n \)s.

\[
\begin{align*}
  f(x) &= e^x \quad \rightarrow \quad c_0 = \frac{f(0)}{0!} = \frac{1}{0!} \\
  f'(x) &= e^x \quad \rightarrow \quad c_1 = \frac{f'(0)}{1!} = \frac{1}{1!} \\
  f''(x) &= e^x \quad \rightarrow \quad c_2 = \frac{f''(0)}{2!} = \frac{1}{2!} \\
  f'''(x) &= e^x \quad \rightarrow \quad c_3 = \frac{f'''(0)}{3!} = \frac{1}{3!} \\
  f^4(x) &= e^x \quad \rightarrow \quad c_4 = \frac{f^4(0)}{4!} = \frac{1}{4!} \\
  \vdots
\end{align*}
\]

2. Next, we try to come up with a formula for \( c_n \). It appears

\( c_n = \frac{1}{n!} \)

3. Write out the terms of the series

\[
\begin{align*}
  f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots \\
  f(x) &= 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \ldots
\end{align*}
\]

4. Put it back into series notation

\[
f(x) = \sum_{0}^{\infty} \frac{1}{n!} x^n
\]
5. Next, let’s find the radius of convergence. We do this by using the Ratio Test.

\[ L = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{x!}{x^n} \right| = \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = 0 \]

for all \( x \).

Since \( L = 0 < 1 \), the Ratio Test concludes the radius of convergence is \( R = \infty \) and the interval of convergence is \((-\infty, \infty)\).

The following statement may sound a bit strange, but here it goes. If \( e^x \) has a power series representation, the above work shows us it must be \( e^x = \sum_{0}^{\infty} \frac{x^n}{n!} \). BUT... we still have to answer this question, “Does \( e^x \) actually have a power series representation?”

Let’s consider the \( n \)-th degree Taylor Polynomial

\[ T_n(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!} (x-a)^i \]

For example, for \( e^x = \sum_{0}^{\infty} \frac{1}{n!} x^n \)

\[ T_1(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!} (x-a)^i = 1 + x \]

\[ T_2(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!} (x-a)^i = 1 + x + \frac{1}{2!} x^2 \]

\[ T_3(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!} (x-a)^i = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 \]

\[ T_4(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!} (x-a)^i = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 \]

So \( T_n(x) \) is a polynomial approximation to \( f(x) \). If we were to let \( n \to \infty \), then

\[ f(x) = \lim_{n \to \infty} T_n(x) = \sum_{0}^{\infty} \frac{1}{n!} x^n \]
Just like when we did partial sums for a series, if we only go out \( n \) terms of the series, we have some remainder.

\[
f(x) = T_n(x) + R_n(x)
\]

where \( R_n(x) \) is called the **Remainder** of the Taylor Series. If we can show

\[
\lim_{n \to \infty} R_n(x) = 0
\]

then this proves \( \sum_{0}^{\infty} \frac{1}{n!} x^n \) is really the power series representation for \( f(x) = e^x \). We prove this by the following theorem,

**Theorem 5.9.** If \( f(x) = T_n(x) + R_n(x) \), where \( T_n(x) \) is the \( n \)-th degree Taylor polynomial of \( f \) at \( x = a \) and

\[
\lim_{n \to \infty} R_n(x) = 0
\]

for \( |x - a| < R \), then \( f \) is equal to the sum of its Taylor Series on the interval \( |x - a| < R \).

In order to show \( \lim_{n \to \infty} R_n(x) = 0 \), we use the following theorem

**Theorem 5.10 (Taylors Inequality).** If \( |f^{n+1}(x)| \leq M \) for \( |x - a| \leq d \), then the remainder \( R_n(x) \) of the Taylor Series satisfies the inequality

\[
|R_n(x)| \leq \frac{M}{(n + 1)!} |x - a|^{n+1}
\]

for \( |x - a| < d \).

Our goal is to find \( M \), take the limit of \( \frac{M}{(n + 1)!} |x - a|^{n+1} \) as \( n \to \infty \) and hope it goes to 0. This will force \( R_n(x) \to 0 \).
1. Think of $M$ as the largest value $f^{n+1}(x)$ can take on over its interval $|x - 0| < d$. For our function $f^{n+1}(x) = e^x$, it will take on its largest value at $x = d$.

\[ e^x \leq e^d \text{ on the interval } -d < x < d \]

2. So let $M = e^d$. Note, $a = 0$

\[ |R_n(x)| \leq \frac{e^d}{(n + 1)!}|x|^{n+1} \]

and

\[ \lim_{n \to \infty} \frac{e^d}{(n + 1)!}|x|^{n+1} = 0 \text{ for all } x \]

3. Therefore, $R_n(x) \to 0$. This finally shows that

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

4. Also, if $x = 1$, we get

\[ e^1 = e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots \]

**Example 5.68.** Suppose we wanted to find the Taylor Series for $f(x) = e^x$ at $a = 2$. 

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1. First, organize a table to find $c_n$

\[
\begin{align*}
  f(x) &= e^x \quad \rightarrow \quad c_0 = \frac{f(2)}{0!} = \frac{e^2}{0!} \\
  f'(x) &= e^x \quad \rightarrow \quad c_1 = \frac{f'(2)}{1!} = \frac{e^2}{1!} \\
  f''(x) &= e^x \quad \rightarrow \quad c_2 = \frac{f''(2)}{2!} = \frac{e^2}{2!} \\
  f'''(x) &= e^x \quad \rightarrow \quad c_3 = \frac{f'''(2)}{3!} = \frac{e^2}{3!} \\
  f^4(x) &= e^x \quad \rightarrow \quad c_4 = \frac{f^4(2)}{4!} = \frac{e^2}{4!} \\
  &\vdots
\end{align*}
\]

2. Based on the above table, $c_n = \frac{e^2}{n!}$.

3. So $f(x) = e^x = \sum_{0}^{\infty} c_n(x - 2)^n = \sum_{0}^{\infty} \frac{e^2}{n!}$

**Example 5.69.** Let’s find the Maclaurin series for $\cos(x)$

1. First, organize a table to find $c_n$. 


\[
f(x) = \cos(x) \quad \rightarrow \quad \frac{f(0)}{0!} = 1
\]
\[
f'(x) = -\sin(x) \quad \rightarrow \quad \frac{f'(0)}{1!} = 0
\]
\[
f''(x) = -\cos(x) \quad \rightarrow \quad \frac{f''(0)}{2!} = -\frac{1}{2!}
\]
\[
f'''(x) = \sin(x) \quad \rightarrow \quad \frac{f'''(0)}{3!} = 0
\]
\[
f''''(x) = \cos(x) \quad \rightarrow \quad \frac{f''''(0)}{4!} = 1
\]
\[
f'''''(x) = -\sin(x) \quad \rightarrow \quad \frac{f'''''(0)}{5!} = 0
\]
\[
f''''''(x) = -\cos(x) \quad \rightarrow \quad \frac{f''''''(0)}{6!} = -\frac{1}{6!}
\]
\[
f'''''''(x) = \sin(x) \quad \rightarrow \quad \frac{f'''''''(0)}{7!} = 0
\]
\[
f''''''''(x) = \cos(x) \quad \rightarrow \quad \frac{f''''''''(0)}{8!} = \frac{1}{8!}
\]
\[
\vdots
\]

2. Well it might be hard to determine a formula for \(c_n\) from here. Let's write out the terms of the series and see if that helps.

\[
\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \ldots
\]

It looks like the degree and the factorial go up by 2. So we can write the series like this

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}
\]

3. To check that this is truly the power series representation, let's use Taylor's Inequality to show \(R_n(x) \rightarrow 0\).

(a) Note that the derivatives alternate between \(\cos(x)\) and \(\sin(x)\). So...
\[ |f^{n+1}| \leq 1 \text{ for all } x \]

(b) This means we set \( M = 1 \).

(c) Now onto Taylor’s Inequality

\[ |R_n(x)| \leq \frac{M}{(n+1)!} |x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!} \]

(d) As \( n \to \infty \),

\[ \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \]

(e) Therefore, \( R_n(x) \to 0 \), and we have

\[
\cos(x) = \sum_{0}^{\infty} \frac{1}{(2n)!} x^{2n}
\]

**Example 5.70.** Use the previous example to find the Maclaurin Series for \( \sin(x) \).

Since \( \sin(x) \) is the anti-derivative of \( \cos(x) \), we just integrate the Maclaurin Series for \( \cos(x) \).

\[
\sin(x) = \int \cos(x) \, dx = \int \sum_{0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \, dx = \sum_{0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+1}.
\]
Example 5.71. How about \( f(x) = x^3 \sin(x) \)?

It’s actually very simple. We already know the power series representation for \( \sin(x) \). Now we just multiply it by \( x^3 \).

\[
x^3 \sin(x) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n + 1)!}
\]

Example 5.72. How about \( f(x) = \cos(\pi x^2) \)

\[
\cos(\pi x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{4n}}{(2n)!}
\]

5.10.1 Important Maclaurin Series to Memorize

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\]

\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

\[
\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n + 1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots
\]

Example 5.73. Estimate \( \int_0^1 e^{-x^2} \, dx \)

1. Find the Maclaurin Series for \( e^{-x^2} \)

\[
e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}
\]
2. Now we integrate the series

\[ \int_0^1 e^{-x^2} \, dx = \sum_{n=0}^{\infty} \frac{2n+1}{(2n+1)n!} \bigg|_0^1 \]

3. Let's write out the first few terms of this series

\[ \int_0^1 e^{-x^2} \, dx = x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \ldots \bigg|_0^1 \]

4. Note that when we plug in \( x = 0 \), we get 0. So we just have to plug in \( x = 1 \).

\[ \int_0^1 e^{-x^2} \, dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \ldots \]

5. Unless you’re given an estimate error, just add up the first few terms.

\[ 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = 0.7575 \]

6. Since this is an alternating series, we can estimate the error. It must be less than the next term of the series.

\[ \text{Error} < \frac{1}{11 \cdot 5!} < 0.000756 \]

So our estimate is off by at most 0.000756.

**Example 5.74.** Find \( T_3(x) \) when \( f(x) = \tan^{-1}(x) \) at \( a = 1 \).
1. Let’s start with a table to determine $c_n$. 

<table>
<thead>
<tr>
<th>$f^n(x)$</th>
<th>$f^n(1)$</th>
<th>$c_n = \frac{f^n(1)}{n!}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = \tan^{-1}(x)$</td>
<td>$f(1) = \pi/4$</td>
<td>$c_0 = \frac{\pi/4}{0!} = \pi/4$</td>
</tr>
<tr>
<td>$f'(x) = \frac{1}{1 + x^2}$</td>
<td>$f'(1) = \frac{1}{2}$</td>
<td>$c_1 = \frac{1/2}{1!} = \frac{1}{2}$</td>
</tr>
<tr>
<td>$f''(x) = -\frac{2x}{(1 + x^2)^2}$</td>
<td>$f''(1) = -\frac{2}{4}$</td>
<td>$c_2 = \frac{-2/4}{2!} = -\frac{1}{4}$</td>
</tr>
<tr>
<td>$f'''(x) = \frac{6x^2 - 2}{(x^2 + 1)^3}$</td>
<td>$f'''(1) = \frac{4}{8}$</td>
<td>$c_3 = \frac{4/8}{3!} = \frac{1}{12}$</td>
</tr>
</tbody>
</table>

2. Use $T_3(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3$

3. $T_3(x) = \frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2 + \frac{1}{12}(x - 1)^3$

Example 5.75. Let $f(x) = e^{x^2}$, $a = 0$, on the interval $[0, 0.1]$.

1. Find $T_3(x)$

<table>
<thead>
<tr>
<th>$f^n(x)$</th>
<th>$f^n(0)$</th>
<th>$c_n = \frac{f^n(0)}{n!}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = e^{x^2}$</td>
<td>$f(0) = 1$</td>
<td>$c_0 = \frac{1}{0!} = 1$</td>
</tr>
<tr>
<td>$f'(x) = e^{x^2} \cdot 2x$</td>
<td>$f'(0) = 0$</td>
<td>$c_1 = \frac{0}{1} = 0$</td>
</tr>
<tr>
<td>$f''(x) = e^{x^2} \cdot 2 + 2xe^{x^2} \cdot 2x$</td>
<td>$f''(0) = 2$</td>
<td>$c_2 = \frac{2}{2!} = 1$</td>
</tr>
<tr>
<td>$f'''(x) = e^{x^2}(12x + 8x^3)$</td>
<td>$f'''(0) = 0$</td>
<td>$c_3 = \frac{0}{3!} = 0$</td>
</tr>
</tbody>
</table>

2. Write out $T_3(x) = c_0 + c_1x + c_2x^2 + c_3x^3$

\[T_3(x) = 1 + 0x + 1x^2 + 0x^3 = 1 + x^2\]
3. Find the maximum error over the interval $[0, 0.1]$.

$$|R_3(x)| \leq \frac{M}{4!}|x|^4$$

(a) Let’s bound $|x|^4$

$$0 \leq x \leq 0.1 \rightarrow |x - 0| \leq 0.1$$
$$\rightarrow |x| \leq 0.1$$
$$\rightarrow |x|^4 \leq 0.1^4$$

(b) Let’s bound $M$

$$|f^{(4)}(x)| = |e^{x^2}(12 + 48x^2 + 16x^4)|$$
$$\leq |f^{(4)}(0.1)|$$
$$\leq e^{0.1^2}(12 + .48 + 0.0016)$$

4. Find the error $|R_3(x)|$

$$|R_3(x)| \leq \frac{e^{0.1^2}(12 + .48 + 0.0016)}{4!}(0.1)^4 = 0.00006$$

5. Conclusion: Over the interval $[0, 0.1]$, $f(x) = e^{x^2}$ and $T_3(x) = 1 + x^2$ differ by at most 0.00006.