5.8 Power Series

Consider the following series

\[ \sum c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots \]

where \( c_n \)s are coefficients to \( x_n \).

This is called a power series centered at \( a = 0 \). A general power series, centered at \( a \), is

\[ \sum c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + c_3 (x - a)^3 + c_4 (x - a)^4 + \ldots \]

We have already seen some examples of power series. We will focus on two things during these sections. The first is to determine what values of \( x \) will allow the series to converge. The second is writing certain functions as a series.

Example 5.55. The following geometric series

\[ \sum x^n = 1 + x + x^2 + x^3 + x^4 + \ldots \]

will converge when \(-1 < x < 1\). We also know that a geometric series with radius \( x \) will converge to

\[ \sum x^n = \frac{1}{1-x} \]

Let’s look at some examples of power series and determine what values of \( x \) will allow the series to converge.

Example 5.56.

1. For what values of \( x \) does \( \sum \frac{(x + 7)^n}{n^2 + 1} \) converge?

We do this by using the Ratio or Root test. In this case, we’ll use the Ratio Test.

\[ L = \lim_{n \to \infty} \left| \frac{(x + 7)^{n+1}}{(n + 1)^2 + 2} \cdot \frac{n^2 + 1}{(x + 7)^n} \right| = \lim_{n \to \infty} \left| \frac{(x + 7)}{1} \cdot \frac{n^2 + 1}{(n + 1)^2 + 1} \right| \]
\[ = |x + 7| \lim_{n \to \infty} \left| \frac{n^2 + 1}{(n + 1)^2 + 1} \right| = |x + 7| \cdot 1 = |x + 7| \]

We know by the Ratio Test, the series will converge absolutely when \( L < 1 \). For us, \( L = |x + 7| \). So we solve this inequality to find what values of \( x \) will make \( L < 1 \).

\[
|x + 7| < 1 \\
-1 < x + 7 < 1 \\
-8 < x < -6
\]

So any value between -8 and -6 will make the series converge. But what about \( x = -8 \) or \( x = -6 \)? Do they make the series converge? You need to check the endpoints of your interval individually.

(a) \( x = -8 \)

\[ \sum \frac{(-8 + 7)^n}{n^2 + 1} = \sum \frac{(-1)^n}{n^2 + 1}, \]

which converges by the Alternating Series Test.

(b) \( x = -6 \)

\[ \sum \frac{(-6 + 7)^n}{n^2 + 1} = \sum \frac{1}{n^2 + 1}, \]

which converges by the a Direct Comparison with a \( p \)-series \( \sum \frac{1}{n^2} \).

Keep in mind that the series may not converge at the endpoints. For example, \( \sum \frac{(x + 7)^n}{n + 1} \), will converge at one of the endpoints but not the other. The endpoints will be the same as the previous example, but will not work at \( x = -6 \), since

\[ \sum \frac{(-6 + 7)^n}{n + 1} = \sum \frac{1}{n + 1} \text{ diverges} \]

The lesson is, never assume the series will converge at the endpoints.
2. Find all \( x \)-values that allow \( \sum_{n=1}^{\infty} n!x^n \n\)

Let’s use the Ratio Test to find \( L \).

\[
L = \lim_{n \to \infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n \to \infty} \frac{(n+1)n!x^n x}{n!x^n} = \lim_{n \to \infty} (n+1)x = \infty
\]

Notice that the \( x \) is insignificant. As \( n \to \infty \), so does \( L \). There is only one \( x \) value that will allow this series to converge and that’s \( x = 0 \). Therefore, the series \( \sum_{n=1}^{\infty} n!x^n \) converges only when \( x = 0 \).

3. Find all \( x \)-values that allow \( \sum (-1)^{n}x^{2n} \frac{(2n)!}{(2n)!} \) to converge.

Again, we use the Ratio Test to find \( L \).

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1}x^{2(n+1)}(2n)!}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^{n}x^{2n}} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^{n}x^{2n}} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{(-1)^{n+1}x^{2n}x^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(-1)^{n}x^{2n}} \right| = \lim_{n \to \infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0
\]

for all \( x \).

In this example, it doesn’t matter what \( x \) is since the limit will approach 0. Therefore, \( \sum (-1)^{n}x^{2n} \frac{(2n)!}{(2n)!} \) converges for all \( x \) or \( (-\infty, \infty) \).

There are three possible types of convergence for a power series \( \sum c_n(x-a)^n \). The previous three examples demonstrate the three types of convergence. The three types are

1. \( \sum c_n(x-a)^n \) converges only when \( x = a \), as in example (2)

2. \( \sum c_n(x-a)^n \) converges for all \( x \), as in example (3)
3. There is a positive $R$ such that if $|x - a| < R$, $\sum c_n(x - a)^n$ converges. If $|x - a| > R$, then $\sum c_n(x - a)^n$ diverges. This is what happens in example (1) above. The number $R$ is called the **Radius of Convergence**. The intervals of convergence will be centered around $x = a$. You will have to check the endpoints of the interval of convergence separately to determine if the series converges.

**Example 5.57.** Determine the Radius and Interval of convergence for the following series:

1. $\sum \frac{(-1)^n x^n}{4^n \ln n}$

   As always, use the Ratio Test to find $L$.

   \[
   L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{n+1} 4^n \ln n}{(-1)^n x^n 4^{n+1} \ln(n+1)} \right| = \lim_{n \to \infty} \left| \frac{4^n}{4^{n+1}} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{\ln n}{\ln(n+1)} \right| = \left| x \right| \lim_{n \to \infty} \frac{\ln n}{\ln n + 1} = \left| x \right| \cdot \frac{1}{4}
   \]

   So our series will converge when

   \[
   L = \left| \frac{x}{4} \right| < 1
   \]

   \[
   \left| x \right| < 4
   \]

   We can see the Radius of convergence $R$ is 4. Solving this inequality, we get $-4 < x < 4$. Now we check the endpoints.

   (a) $x = -4$

   \[
   \sum \frac{(-1)^n (-4)^n}{4^n \ln n} = \sum \frac{4^n}{4^n \ln n} = \sum \frac{1}{\ln n}
   \]

   which diverges. We can check the divergence by comparing it to the $p$-series $\sum \frac{1}{n}$. 

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(b) \( x = 4 \)

\[
\sum (-1)^n(4)^n = \sum (-1)^n \frac{4^n \ln n}{n} \]

which converges by the Alternating Series Test.

Therefore, \( \sum \frac{(-1)^n x^n}{4^n \ln n} \) converges on the interval \((-4, 4]\).

2. \( \sum \frac{n(5x + 2)^n}{3^{n+1}} \)

(a) As always, use the Ratio Test

\[
L = \lim_{n \to \infty} \left| \frac{(n + 1)(5x + 2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(5x + 2)^n} \right| = \lim_{n \to \infty} \left| \frac{n + 1}{n} \cdot \frac{5x + 2}{3} \right| = \left| \frac{5x + 2}{3} \right|
\]

(b) \( L = \left| \frac{5x + 2}{3} \right| \) will converge when

\[
\left| \frac{5x + 2}{3} \right| < 1
\]

Before solving for \( x \) to find the interval of convergence, I want to find the center \( a \) and the radius \( R \). We can do it from this point, as long as its in the form \( |x - a| \)

\[
\left| \frac{5}{3} \cdot \left( x + \frac{2}{5} \right) \right| < 1
\]

\[
\left| x + \frac{2}{5} \right| < \frac{3}{5}
\]

From here, we see the radius of convergence is \( R = \frac{3}{5} \) with the center \( a = -\frac{2}{5} \).
(c) Let’s keep solving for $x$ to find the endpoints.

\[-\frac{3}{5} < x + \frac{2}{5} < \frac{3}{5}\]

\[-1 < x < \frac{1}{5}\]

(d) Now we check the endpoints, $x = -1$ and $x = \frac{1}{5}$

i. $x = -1$

\[\sum \frac{n(5(-1) + 2)^n}{3^{n+1}} = \sum \frac{n(-3)^n}{3(3)^n} = \sum \frac{(-1)^n n}{3}\]

which diverges by the Test for Divergence.

ii. $x = \frac{1}{5}$

\[\sum \frac{n(5(1/5) + 2)^n}{3^{n+1}} = \sum \frac{n(3)^n}{3(3)^n} = \sum \frac{n}{3} \to \infty\]

Therefore, \[\sum \frac{n(5x + 2)^n}{3^{n+1}}\] converges on the interval \((-1, \frac{1}{5})\).