Arc Length

Recall the length of a line segment:

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

But what about something like this? How can we find how long this curve is?
1. The idea is to divide the interval into \( n \) equal subintervals each with width \( \Delta x \).

2. Find the length between the lines connecting the \( y \)-values \((P_1P_2, P_2P_3, \text{etc.})\).

3. The length of the curve in each subinterval is approximately the length of each line segment.

4. Add all those lengths together to get an approximate arc length.

\[
L \approx \sum_{i=1}^{n} P_{i-1}P_i
\]

5. The larger \( n \) gets, the better the approximation.

\[
L = \lim_{n \to \infty} \sum_{i=1}^{n} P_{i-1}P_i
\]
So how do we go from a limit to using an integral to calculate arc length.

Consider the following picture that contains the length of one segment.

1. Find the length again

\[ d = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x_i^2 + \Delta y_i^2} \]

2. By the Mean Value Theorem, we know the interval \([x_{i-1}, x_i]\), there is a point \(x_i^*\) such that

\[ f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}) \]
\[ \Delta y_i = f'(x_i^*)\Delta x_i \]

We can now write the length of the line segment as

\[
P_{i-1}P_i = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}
= \sqrt{\Delta x_i^2 + \Delta y_i^2}
= \sqrt{\Delta x_i^2 + [f'(x_i^*)]^2 \Delta x_i^2}
= \sqrt{\Delta x_i^2 (1 + [f'(x_i^*)]^2)}
= \sqrt{1 + [f'(x_i^*)]^2} \cdot \Delta x_i
\]
3. The total length,

\[ L = \lim_{n \to \infty} \sum_{i=0}^{n} \sqrt{1 + f'(x_i)^2} \cdot \Delta x \]

which we rewrite as

\[ L = \int_{a}^{b} \sqrt{1 + f'(x)^2} \, dx \]

or

\[ L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \]

If you’re given a function in terms of \( y \), \( x = h(y) \), the formula would be

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**Formula 1: Arc Length Formula**

\[
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{if in terms of } x \\
L = \int_{c}^{d} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \quad \text{if in terms of } y
\]

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**Example 1**

Find the length of the arc of the semicubical parabola \( y^2 = x^3 \) between the points (1,1) and (4,8).

1. To graph it, rewrite it as \( y = x^{3/2} \).
2. Find \( \frac{dy}{dx} \)

\[
\frac{dy}{dx} = \frac{2}{3} x^{1/2}
\]

3. Use the Arc Length Formula

\[
L = \int_{1}^{4} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

\[
= \int_{1}^{4} \sqrt{1 + \frac{9}{4} x} \, dx
\]

We need to use substitution to finish integrating.

(a) Let \( u = 1 + \frac{9}{4} x \)

(b) \( du = \frac{9}{4} x \rightarrow \frac{4}{9} du = dx \)

(c) Change the limits of integration

If \( x = 1, u = 13/4 \)

If \( x = 4, u = 10 \)

(d) Integrate

\[
\int_{1}^{4} \sqrt{1 + \frac{9}{4} x} \, dx = \frac{4}{9} \int_{13/4}^{10} u^{1/2} \, du
\]

\[
= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \bigg|_{13/4}^{10}
\]

\[
= 7.63371
\]

Example 2

Find the exact length of the curve \( x = \frac{1}{3} \sqrt{y(y - 3)} \) over the interval \([1,9]\).
1. This is a function in terms of $y$. In order to graph this, you’ll have to use the techniques from Chapter 1.

2. The graph

3. Find $\frac{dx}{dy}$

\[
\frac{dx}{dy} = \frac{1}{3} y^{1/2} \cdot (1) + (y - 3) \cdot \frac{1}{6} y^{-1/2} \\
= \frac{1}{3} y^{1/2} + \frac{1}{6 \sqrt{y}} (y - 3) \\
= \frac{1}{6 \sqrt{y}} (2y + (y - 3)) \\
= \frac{1}{6 \sqrt{y}} (3y - 3) \\
= \frac{(y - 1)}{2 \sqrt{y}}
\]
4. Find \(1 + \left( \frac{dx}{dy} \right)^2\)

\[
1 + \left( \frac{y - 1}{2\sqrt{y}} \right)^2 = 1 + \frac{(y - 1)^2}{4y} \\
= \frac{4y + y^2 - 2y + 1}{4y} \\
= \frac{y^2 + 2y + 1}{4y} \\
= \frac{(y + 1)^2}{4y}
\]

5. Set up the integral

\[
L = \int_1^9 \sqrt{1 + \left( \frac{y - 3}{2\sqrt{y}} \right)^2} \, dy \\
= \int_1^9 \sqrt{\frac{(y + 1)^2}{4y}} \, dy \\
= \int_1^9 \frac{y + 1}{2y^{1/2}} \, dy \\
= \int_1^9 \frac{1}{2} y^{1/2} + \frac{1}{2} y^{-1/2} \, dy \\
= \frac{1}{3} y^{3/2} + y^{1/2} \bigg|_1^9 \\
= (9 + 3) - \left( \frac{1}{3} + 1 \right) \\
= 10.6667
\]