Improper Integrals

Well you’ve made it through all the integration techniques. Congrats! Unfortunately for us, we still need to cover one more integral. They are called Improper Integrals.

At this point, we’ve only dealt with integrals of the form

\[ \int_{a}^{b} f(x) \, dx \]

Before we talk about the improper type, let’s try to build up to it.

Consider the integral

\[ \int_{1}^{t} \frac{1}{x^2} \, dx \]

Evaluating the integral, we get

\[ \int_{1}^{t} \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_{1}^{t} = -\frac{1}{t} + 1 = 1 - \frac{1}{t} \]

Does this work for any \( t > 1 \)?
1. \[ \int_1^2 \frac{1}{x^2} \, dx = -\frac{1}{x}\bigg|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2} \]

2. \[ \int_1^{10} \frac{1}{x^2} \, dx = -\frac{1}{x}\bigg|_1^{10} = -\frac{1}{10} + 1 = \frac{9}{10} \]

3. \[ \int_1^{100} \frac{1}{x^2} \, dx = -\frac{1}{x}\bigg|_1^{100} = -\frac{1}{100} + 1 = \frac{99}{100} \]

4. \[ \int_1^{1000} \frac{1}{x^2} \, dx = -\frac{1}{x}\bigg|_1^{1000} = -\frac{1}{1000} + 1 = \frac{999}{1000} \]

5. \[ \int_1^{50000} \frac{1}{x^2} \, dx = -\frac{1}{x}\bigg|_1^{50000} = -\frac{1}{50000} + 1 = \frac{49999}{50000} \]

It appears when \( b \) gets very large, the value of the integral approaches 1. Notice that

\[ \lim_{t \to \infty} -\frac{1}{t} + 1 = 1 \]

What this says is as \( t \) gets very large, say

\[ \int_1^{100000000} \frac{1}{x^2} \, dx \]

then the value of that integral is extremely close to 1. And as \( t \to \infty \), the value of the integral converges to 1.
Definition 1: Definition of an Improper Integral, Type I

1. If \( \int_a^t f(x) \, dx \) exists for every number \( t \geq a \), then

\[
\int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx
\]

2. If \( \int_t^b f(x) \, dx \) exists for every number \( t \leq b \), then

\[
\int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx
\]

3. If \( \int_a^\infty f(x) \, dx \) or \( \int_{-\infty}^b f(x) \, dx \) exist, they are called **convergent**. If they do not exist, we call them **divergent**.
Example 1

Find $\int_1^\infty \frac{1}{x} \, dx$

Recall, we saw $\int_1^\infty \frac{1}{x^2} \, dx = 1$. Let’s see about $\frac{1}{x}$.

\[
\int_1^\infty \frac{1}{x} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x} \, dx = \lim_{t \to \infty} \ln(x)|_1^t = \lim_{t \to \infty} \ln(t) - \ln(1) = \lim_{t \to \infty} \ln(t) = \infty
\]

This is interesting. We changed the degree of the denominator slightly, and now the integral diverges. The natural question at this point would be, what does $p$ have to be so

\[
\int_1^\infty \frac{1}{x^p} \, dx \text{ exists?}
\]

I’m warning you ahead of time that we do a bit of algebra hocus pocus here. Nothing too bad though.
\[
\int_1^\infty \frac{1}{x^p} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^p} \, dx \\
= \lim_{t \to \infty} \int_1^t x^{-p} \, dx \\
= \lim_{t \to \infty} \left[ \frac{x^{-p+1}}{-p+1} \right]_1^t \\
= \lim_{t \to \infty} \frac{1}{1-p} \cdot \frac{1}{x^{p-1}} \bigg|_1^t \\
= \lim_{t \to \infty} \frac{1}{1-p} \left[ \frac{1}{t^{p-1}} - 1 \right]
\]

If \( p > 1 \), then \( p - 1 > 0 \), so \( t^{p-1} \to \infty \). Therefore, we have \( \frac{1}{t^{p-1}} \to 0 \).

\[
\int_1^\infty \frac{1}{x^p} \, dx = \frac{1}{p-1}
\]

What happens when \( p < 1 \)? We already know what happens when \( p = 1 \).

If \( p < 1 \), then \( p - 1 < 0 \). This makes \( t^{p-1} \to 0 \) as \( t \to \infty \). Therefore, \( \frac{1}{t^{p-1}} \to \infty \) which makes

\[
\int_1^\infty \frac{1}{x^p} \, dx = \infty \text{ (divergent)}
\]

If you don’t see why \( t^{p-1} \to \infty \) when \( t \to \infty \), give yourself an example. Let \( p = 0.5 \).

\[
\lim_{t \to \infty} t^{0.5-1} = \lim_{t \to \infty} t^{-1/2} = \lim_{t \to \infty} \frac{1}{t^{1/2}} = 0
\]

Which means,

\[
\lim_{t \to \infty} \frac{1}{t^{0.5}} = \lim_{t \to \infty} t^{0.5} = \infty
\]
This particular integral will be extremely important for the rest of the semester. So memorize it!

Example 2

\[
\int_{-\infty}^{0} xe^x \, dx
\]

Note, all of these integrals will have limits. Many of them will require L’Hospitals Rule.

\[
\int_{-\infty}^{0} xe^x \, dx = \lim_{t \to -\infty} \int_{t}^{0} xe^x \, dx
\]

And...this one requires Integration by Parts.

1. Let \( u = x \), \( dv = e^x \, dx \)

2. So \( du = dx \), \( v = e^x \)

\[
\lim_{t \to -\infty} \int_{t}^{0} xe^x \, dx = \lim_{t \to -\infty} xe^x - \int_{t}^{0} e^x \, dx
\]
\[
= \lim_{t \to -\infty} xe^x - e^x \bigg|_{t}^{0}
\]
\[
= \lim_{t \to -\infty} (-0e^0 - e^0) - (te^t - e^t)
\]
\[
= \lim_{t \to -\infty} e^t - te^t - 1
\]

3. We know \( \lim_{t \to -\infty} e^t = 0 \)

4. We need to find \( \lim_{t \to -\infty} -te^t \)
\[
\lim_{t \to -\infty} -te^t = \lim_{t \to -\infty} \frac{-t}{e^{-t}}
\]
\[
\overset{LH}{\Rightarrow} \lim_{t \to -\infty} -1 = -\infty
\]
\[
= 0
\]

5. So our final answer is

\[
\int_{-\infty}^{0} xe^x \, dx = \lim_{t \to -\infty} e^t - te^t - 1 = -1
\]

Seems like we’re going to have fun in this section!

**Example 3**

\[
\int_{1}^{\infty} \frac{x + 1}{x^2 + 2x} \, dx.
\]

Let’s set it up with the limit notation.

\[
\int_{1}^{\infty} \frac{x + 1}{x^2 + 2x} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{x + 1}{x^2 + 2x} \, dx
\]

For now, we’ll drop the \( \lim_{t \to \infty} \) until after we anti-differentiate. Also, the integral requires a basic \( u \)-substitution.

1. Let \( u = x^2 + 2x \)

2. \( du = 2x + 2 \, dx \to \frac{1}{2} \, du = x + 1 \, dx \)

3. Change the bounds
If \( x = t \), then \( u = t^2 + 2t \)

If \( x = 1 \), then \( u = 3 \)

4. Substitute!

\[
\int_1^t \frac{x + 1}{x^2 + 2x} \, dx = \frac{1}{2} \int_3^{t^2 + 2t} \frac{1}{u} \, du
\]

\[
= \frac{1}{2} \ln |u|_{3}^{t^2 + 2t}
\]

Let’s get that limit back in.

\[
\lim_{t \to \infty} \int_3^{t^2 + 2t} \frac{1}{u} \, du = \lim_{t \to \infty} \ln |u|_{3}^{t^2 + 2t}
\]

\[
= \lim_{t \to \infty} \ln |t^2 + 2t| - \ln |3|
\]

\[
= \infty - \ln |3|
\]

\[
= \infty
\]

**Example 4**

\[
\int_0^{\infty} \frac{dx}{(x + 3)(x + 4)}
\]

We need to use partial fractions.

\[
\frac{1}{(x + 3)(x + 4)} = \frac{A}{x + 3} + \frac{B}{x + 4}
\]

1. Start by multiply both sides by \((x + 3)(x + 4)\)

\[
1 = A(x + 4) + B(x + 3)
\]
2. Distribute and collect like terms

\[ 1 = (A + B)x + (4A + 3B) \]

3. Match the coefficients

\[ A + B = 0 \]
\[ 4A + 3B = 1 \]

4. Solving this system, we get \( A = 1 \) and \( B = -1 \)

5. Rewrite the integral

\[
\int_0^\infty \frac{1}{x + 3} - \frac{1}{x + 4} \, dx = \lim_{t \to \infty} \int_0^t \frac{1}{x + 3} - \frac{1}{x + 4} \, dx
\]

\[
= \lim_{t \to \infty} \left[ \ln |x + 3| - \ln |x + 4| \right]_0^t
\]

\[
= \lim_{t \to \infty} \ln \left| \frac{x + 3}{x + 4} \right|_0^t
\]

\[
= \lim_{t \to \infty} \ln \left| \frac{t + 3}{t + 4} - \ln \frac{3}{4} \right|
\]

\[
= 0 - \ln \left| \frac{3}{4} \right|
\]

\[
= \ln \left| \frac{4}{3} \right|
\]

---

**Example 5**

\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx \]

We don’t know how to do this when both bounds are \( \pm \infty \). Recall the following property of integrals
\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx
\]

where \(a < c < b\)

We use this property now. Choose a number (a nice one) between \((-\infty, \infty)\). How about 0? Rewrite the integral

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1 + x^2} \, dx + \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx
\]

Let’s do each integral separately.

1. \(\int_{-\infty}^{0} \frac{1}{1 + x^2} \, dx\)

\[
\int_{-\infty}^{0} \frac{1}{1 + x^2} \, dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1 + x^2} \, dx
\]
\[
= \lim_{t \to -\infty} \tan^{-1}(x) \bigg|_{t}^{0}
\]
\[
= \lim_{t \to -\infty} \tan^{-1}(0) - \tan^{-1}(t)
\]
\[
= 0 - (-\pi/2)
\]
\[
= \pi/2
\]

2. \(\int_{0}^{\infty} \frac{1}{1 + x^2} \, dx\)

\[
\int_{0}^{\infty} \frac{1}{1 + x^2} \, dx = \lim_{t \to \infty} \int_{0}^{t} \frac{1}{1 + x^2} \, dx
\]
\[
= \lim_{t \to \infty} \tan^{-1}(x) \bigg|_{0}^{t}
\]
\[
= \lim_{t \to \infty} \tan^{-1}(t) - \tan^{-1}(0)
\]
\[
= \pi/2 - 0
\]
\[
= \pi/2
\]
3. Final Answer

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi
\]

We conclude the type of integral where \( \infty \) is a bound. Now we move on to the second type of improper integrals.
**Definition 2: Type 2 Improper Integrals**

This type of improper integral involves integrals where a bound is where a vertical asymptote occurs, or when one exists in the interval.

1. If \( f \) is continuous at \([a, b)\) but discontinuous at \( b \), then

\[
\int_{a}^{b} f(x) \, dx = \lim_{t \to b^-} \int_{a}^{t} f(x) \, dx
\]

2. If \( f \) is continuous at \((a, b]\) but discontinuous at \( a \), then

\[
\int_{a}^{b} f(x) \, dx = \lim_{t \to a^+} \int_{t}^{b} f(x) \, dx
\]

3. If \( f \) has a discontinuity at \( x = c \), where \( a < c < b \) then

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx
\]
Example 6

Find \( \int_{0}^{1} \frac{dx}{\sqrt{1-x^2}} \)

This integral is improper because we have an infinite discontinuity (asymptote) at \( x = 1 \). So we use (1) from above.

\[
\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{t \to 1^-} \int_{0}^{t} \frac{dx}{\sqrt{1-x^2}} \]
\[
= \lim_{t \to 1^-} \sin^{-1}(x) \bigg|_{0}^{t}
\]
\[
= \lim_{t \to 1^-} \sin^{-1}(t) - \sin^{-1}(0)
\]
\[
= \frac{\pi}{2} - 0
\]
\[
= \frac{\pi}{2}
\]

You can see the process is very much like the previous integrals.

Example 7

Find \( \int_{1}^{2} \frac{dx}{x \ln x} \)

We’ll start by using a \( u \)-substitution. Let \( u = \ln x \), which gives us \( du = \frac{1}{x} \, dx \). Since the integral has bounds, we’ll do the change of bounds now.

\[
u(1) = \ln(1) = 0
\]
\[
u(2) = \ln(2)
\]

Let’s get started with the integration.
\[
\int_{1}^{2} \frac{dx}{x \ln x} = \int_{0}^{\ln 2} \frac{1}{u} \, du \\
= \lim_{t \to 0^+} \int_{t}^{\ln 2} \frac{1}{u} \, du \\
= \lim_{t \to 0^+} \ln |u|_{t}^{\ln 2} \\
= \lim_{t \to 0^+} \ln |\ln 2| - \ln t \\
= \ln 2 - (-\infty) \\
= \infty
\]

Example 8

Find \( \int_{-1}^{2} x^{-2} \, dx \)

Suppose you didn’t check to see if there were any infinite discontinuities. Let’s see what we get.

\[
\int_{-1}^{2} x^{-2} \, dx = \frac{-1}{x}_{-1}^{2} \\
= -\frac{1}{2} + \frac{1}{-1} \\
= -\frac{3}{2}
\]

However, if you use the appropriate improper integral method, you’ll find this is incorrect. You start by finding the discontinuity, which exists at \( x = 0 \). You then break it up into two separate integrals.

\[
\int_{-1}^{2} x^{-2} \, dx = \int_{-1}^{0} x^{-2} \, dx + \int_{0}^{2} x^{-2} \, dx
\]

You then integrate each on its own.

1. Find

\[
\int_{-1}^{0} x^{-2} \, dx
\]
\[
\int_{-1}^{0} x^{-2} \, dx = \left. -\frac{1}{x} \right|_{-1}^{0} = \lim_{t \to 0^-} -\frac{1}{x} \bigg|_{-1}^{t} = \lim_{t \to 0^-} -\frac{1}{t} + \frac{1}{-1} = \infty
\]

2. Find

\[
\int_{0}^{2} x^{-2} \, dx
\]

\[
\int_{0}^{2} x^{-2} \, dx = \lim_{t \to 0^+} -\frac{1}{x} \bigg|_{t}^{2} = \lim_{t \to 0^+} -\frac{1}{2} + \frac{1}{t} = \infty
\]

This shows

\[
\int_{-1}^{0} x^{-2} \, dx = \infty
\]

which is not \(-3/2\).

Now suppose you wanted to integrate

\[
\int_{1}^{\infty} \frac{dx}{x + e^{3x}}
\]

You’ll find that is has no anti-derivative. So we can’t evaluate this in a closed form (i.e., we would have to approximate). The next question is, does this converge to a number? In
other words, does the integral exist? If you think of the integral as area, is the area finite?

**Definition 3: Comparison Test**

Suppose $f$ and $g$ are continuous functions where $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x) \, dx$ is convergent, then $\int_a^\infty g(x) \, dx$ is also convergent.

2. If $\int_a^\infty g(x) \, dx$ diverges, then $\int_a^\infty f(x) \, dx$ also diverges.

The Comparison Test is also valid for improper integrals with infinite discontinuities at the endpoints.

Let’s get back to

$$\int_1^\infty \frac{dx}{x + e^{3x}}$$

The downside to the comparison test is you have to some idea if it’s going to converge or diverge. This helps figure out if you’re suppose to find a nice divergent function or a nice convergent function. Identifying convergent integrals just takes time.

I’m going to guess this integral converges. So I need to find a nice function that’s bigger than $\frac{1}{x + e^{3x}}$, whose integral converges. Note that if I can make the denominator ”smaller,” it makes the whole function larger.

$$x + e^{3x} \geq e^{3x} \text{ when } x \geq 1$$

Therefore,

$$\frac{1}{x + e^{3x}} \leq \frac{1}{e^{3x}}$$
So if I can show $\int_1^{\infty} \frac{1}{e^{3x}} \, dx$ converges, and show $\frac{1}{x + e^{3x}} \geq 0$ for all $x \geq 1$ (which I don’t think needs explaining), then we’re good.

$$\int_1^{\infty} e^{-3x} \, dx = \lim_{t \to \infty} \int_1^{t} e^{-3x} \, dx$$

$$= \lim_{t \to \infty} \left. -\frac{1}{3}e^{-3x} \right|_1^t$$

$$= \lim_{t \to \infty} \left( -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{-3} \right)$$

$$= 0 + \frac{1}{3}e^{-3}$$

Since $\frac{1}{e^{3x}} \geq \frac{1}{x + e^{3x}} \geq 0$ and $\int_1^{\infty} \frac{1}{e^{3x}} \, dx$ converges, then

$$\int_1^{\infty} \frac{1}{x + e^{3x}} \, dx$$ converges