#1 TWO EQUATIONS BUT THREE VARIABLES. THIS MEANS THERE IS NOT ENOUGH INFO TO SOLVE FOR A VARIABLE. IN THIS CASE z

$$\boxed{z = \text{ANY VALUE}}$$

FROM HERE WE CAN CONVERT OUR EQUATIONS TO

$$x+y+z=1 \rightarrow x=1-y-z$$

PLUG $x=1-y-z$ INTO $2x+3y+4z=5$ TO GET

$$2(1-y-z)+3y+4z=5$$

$$2-2y-2z+3y+4z=5$$

$$y+2z+2=5$$

$$\boxed{y=3-2z}$$

FINALLY, PLUG $y=3-2z$ INTO $x=1-y-z$

$$x=1-(3-2z)-z$$

$$\boxed{x=-2+z}$$

FINAL ANSWER: (b) $z=$ ANY VALUE,

$y=3-2z$

$x=z-2$

#2 SOLVE

$$
\begin{aligned}
2x + 3y - 5z &= -14 \\
3x - 2y + 3z &= 17 \\
4x + 3y - 2z &= -1
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} 2 & 3 & -5 & -14 \\ 3 & -2 & 3 & 17 \\ 4 & 3 & -2 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 3/2 & -5/2 & -7 \\ 3 & -2 & 3 & 17 \\ 4 & 3 & -2 & -1 \end{array}\right]
$$

$$
\begin{array}{l} R_2 - 3R_1 \\ R_3 - 4R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 3/2 & -5/2 & -7 \\ 0 & -13/2 & 21/2 & 38 \\ 0 & -3 & 8 & 27 \end{array}\right] \quad \frac{2}{-13}R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 3/2 & -5/2 & -7 \\ 0 & 1 & -21/13 & -76/13 \\ 0 & -3 & 8 & 27 \end{array}\right]
$$

$$
\begin{array}{l} R_1 - \frac{3}{2}R_2 \\ R_3 + 3R_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/13 & 23/13 \\ 0 & 1 & -21/13 & -76/13 \\ 0 & 0 & 41/13 & 123/13 \end{array}\right] \rightarrow \frac{13}{41}R_3 \left[\begin{array}{ccc|c} 1 & 0 & -1/13 & 23/13 \\ 0 & 1 & -21/13 & -76/13 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

$$
\begin{array}{l} R_1 + \frac{1}{13}R_3 \\ R_2 + \frac{21}{13}R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{aligned} x &= 2 \\ y &= -1 \\ z &= 3 \end{aligned}
$$

ANWSER: (d)

#4 FIND BOTTOM LEFT ENTRY

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\Rightarrow \quad a_{11} = 1(5) + 2(7) = 19$$

$$a_{12} = 1(6) + 2(8) = 22$$

$$a_{21} = 3(5) + 4(7) = 43$$

$$a_{22} = 3(6) + 4(8) = 50$$

ANWSER: (C) 43

#5 SOLVE

$$x + y - 2z = 3$$
$$2x - 3y + 3z = 2$$
$$5x - 5y + 4z = 6$$

SET UP AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 2 & -3 & 3 & 2 \\ 5 & -5 & 4 & 6 \end{array}\right]$$

AFTER ROW REDUCING YOU GET

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/5 & 11/5 \\ 0 & 1 & -7/5 & 4/5 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

LAST ROW $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -1 \end{array}\right]$ MEANS

$$0x + 0y + 0z = -1$$

WHICH HAS NO SOLUTION

ANSWER: (e) SYSTEM IS INCONSISTENT

#6 FIND INVERSE TO $A = \begin{pmatrix} 0 & 0 & 4 \\ 2 & 0 & 10 \\ 0 & 4 & 8 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 4 & 1 & 0 & 0 \\ 2 & 0 & 10 & 0 & 1 & 0 \\ 0 & 4 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{ROWS}]{\text{SWAP}} \left[\begin{array}{ccc|ccc} 2 & 0 & 10 & 0 & 1 & 0 \\ 0 & 4 & 8 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 1/2 & 0 \\ 0 & 4 & 8 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 1/2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1/4 \\ 0 & 0 & 4 & 1 & 0 & 0 \end{array}\right]$$

$$\frac{1}{4}R_3 \begin{bmatrix} 1 & 0 & 5 & 0 & 1/2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 1/4 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 - 2R_1]{R_1 - 5R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5/4 & 1/2 & 0 \\ 0 & 1 & 0 & -1/2 & 0 & 1/4 \\ 0 & 0 & 1 & 1/4 & 0 & 0 \end{array}\right]$$

$$\underbrace{\begin{matrix} -5/4 & 1/2 & 0 \\ -1/2 & 0 & 1/4 \\ 1/4 & 0 & 0 \end{matrix}}_{A^{-1}}$$

ANWSER: TOP LEFT CORNER OF A^{-1} IS

(c) $-5/4$

#7 WHICH OF THE FOLLOWING HAVE INFINITELY MANY SOLUTIONS

I: z CAN BE ANY VALUE

II 3 EQUATIONS, 4 VARIABLES

$$x = 10 + y - 8z - 3w$$
$$y = 3 + 8z$$
$$w = 1$$

z CAN BE ANY VALUE

III $y - 8z = 3$ SHOWS UP TWICE. THAT MEANS YOUR SYSTEM IS ACTUALLY

$$\begin{cases} x - y + 8z = 7 \\ y - 8z = 3 \end{cases}$$

SO z CAN BE ANYTHING

IV $y - 8z = 3$ AND $y - 8z = 4$ CAN'T HAPPEN

IMPLIES $3 = 4$ (WHICH IS WRONG)
SO IV HAS NO SOLUTION

FINAL ANSWER: (C) I, II, III

#10 $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

USE THE SHORTCUT: IF $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ THEN

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

IN OTHER WORDS $ad - bc \neq 0$. IF $ad - bc = 0$ THEN THE INVERSE WOULD NOT EXIST.

$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad ad - bc = (-1)(1) - (1)(1) = -1 - 1 = -2$ ✓

$B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $\quad ad - bc = (1)(1) - (-1)(-1) = 1 - 1 = 0$ BAD

$C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad ad - bc = (0)(0) - (0)(0) = 0$ BAD

$D = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow ad - bc = (0)(1) - (1)(0) = 0 - 0 = 0$ BAD

ONLY A^{-1} EXISTS. ANSWER: (b)

#11 $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 10 & 20 \\ 3 & 7 & 14 & 21 \\ 4 & 9 & 17 & 26 \end{pmatrix}$ $A^{-1} = \begin{pmatrix} -35 & -7 & -30 & 35 \\ 26 & 5 & 20 & -24 \\ -4 & -1 & -2 & 3 \\ -1 & 0 & -1 & 1 \end{pmatrix}$

SOLVE

$$\begin{aligned} x + 2y + 3z + 4w &= 0 \\ 2x + 5y + 10z + 20w &= 1 \\ 3x + 7y + 14z + 21w &= 2 \\ 4x + 9y + 17z + 26w &= 3 \end{aligned}$$

SET UP AS

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 10 & 20 \\ 3 & 7 & 14 & 21 \\ 4 & 9 & 17 & 26 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

SOLUTION TO THE SYSTEM IS $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -35 & -7 & -30 & 35 \\ 26 & 5 & 20 & -24 \\ -4 & -1 & -2 & 3 \\ -1 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = -35(0) - 7(1) - 30(2) + 35(3) = 38$

$y = 26(0) + 5(1) + 20(2) - 24(3) = -27$

$z = -4(0) - 1(1) - 2(2) + 3(3) = 4$

$w = -1(0) + 0(1) - 1(2) + 1(3) = 1$

ANSWER: $w = 1$ (b)

\#15 IF YOU CONVERT THE MATRIX BACK TO LINEAR EQUATIONS WE GET

$$x - 3z = 6 \rightarrow x = 6 + 3z$$

$$y + 2z = 7 \rightarrow y = 7 - 2z$$

SINCE THE LAST ROW ARE ALL 0s

z CAN BE ANY VALUE.

AT THIS POINT THE ANSWER CAN BE (b) OR (d)

FOR (b) CHECK IF $x=0$, $y=11$, $z=2$ IS A SOLUTION

$$x = 6 + 3z = 6 + 3(2) = 12 \neq 0$$

SO THIS CAN'T WORK

FOR (d) CHECK $x = 9$, $y = 5$, $z = 1$

$$x = 6 + 3z = 6 + 3(1) = 9 \checkmark$$

$$y = 7 - 2(1) = 5 \checkmark$$

SO (d) WORKS

#17 $\underset{19\times m}{B}\quad\underset{m\times n}{A}\quad\underset{n\times 7}{B}$

SINCE B MUST HAVE SIZE $19\times m$

AND $n\times 7$

IT MUST BE 19×7

#19 THE SYSTEM CAN BE WRITTEN AS

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 5 & 1 & 8 & -2 \\ -6 & -2 & 1 & 0 \\ -1 & 0 & -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 4 \\ -2 \end{bmatrix}$$

$$A \quad x \quad = \quad B$$

THE SOLUTION IS

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = A^{-1} B$$

$$= \begin{bmatrix} 1 & -8 & -4 & -16 \\ -3 & 25 & 12 & 50 \\ 0 & 2 & 1 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 7 \\ 4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -32 \\ 99 \\ 10 \\ 6 \end{bmatrix}$$

SO $z = 10$ ANSWER: (d)

#2b IF $\begin{bmatrix} 4 & 12 \\ 6 & x \end{bmatrix}$ IS NOT INVERTIBLE, WHAT IS X?

IF $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ IS <u>NOT</u> INVERTIBLE THEN

$$ad - bc = 0$$

SO $4x - 72 = 0$

$$4X = 72$$

$$X = 18$$

ANWSER: (d)

#29 A 4×3 A^T 3×4

B 3×5 B^T 5×3

C 4×5 C^T 5×4

(a) $CB^T + A^T$

$(4\times5)(5\times3) + 3\times4$

$(4\times3) + (3\times4)$ CAN'T HAPPEN

(b) $A^T C + B$

$(3\times4)(4\times5) + (3\times5)$

$(3\times5) + (3\times5)$ ✓

(c) CAN'T SINCE (a) CAN'T WORK

(d) $AB + C^T$

$(4\times3)(3\times5) + (5\times4)$

$(4\times5) + (5\times4)$ CAN'T

(e) CAN'T SINCE (b) WORKS

ANSWER: (b)

#39 SOLVE

$$\begin{cases} x - y + 5z = 13 \\ y - 2z = -7 \\ y + 8z = 33 \end{cases}$$

START
$$\left[\begin{array}{ccc|c} 1 & -1 & 5 & 13 \\ 0 & 1 & -2 & -7 \\ 0 & 1 & 8 & 33 \end{array}\right] \rightarrow \begin{array}{c} R_1 + R_2 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 10 & 40 \end{array}\right]$$

$$\frac{1}{10} R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{array}{c} R_1 - 3R_3 \\ R_2 + 2R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$x = -6, \; y = 1, \; z = 4$

ANSWER: (d)