MATH 210 FINITE MATHEMATICS

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4.2 Linear Programming: Minimization / Dual Problems

Example 1

Suppose you're given the following linear programming problem

$$\text{Minimize } C = -2x - 3y$$

$$\left.\begin{aligned} 5x + 4y &\leq 32 \\ x + 2y &\leq 10 \end{aligned}\right] \text{SPECIAL CASE SINCE THE INEQUALITIES ARE SET UP AS A STANDARD MAX PROBLEM}$$

$$x \geq 0, \qquad y \geq 0$$

TRICK: MINIMIZE C, MAXIMIZE $-C$

MAXIMIZE $-C = \boxed{P = 2x + 3y}$

SUBJECT TO

$5x + 4y \leq 32$

$x + 2y \leq 10$

$x \geq 0, y \geq 0$

INITIAL TABLE ⟶ FINAL TABLE

x	y	S_1	S_2	P	CONST
5	4	1	0	0	32
1	2	0	1	0	10
-2	-3	0	0	1	0

x	y	S_1	S_2	P	CONST
1	0	1/3	-2/3	0	4
0	1	-1/6	5/6	0	3
0	0	1/6	7/6	1	17 ← P

$x = 4, y = 3$, MINIMIUM OF $C = -17$

Definition 1: The Dual Problem for Standard Minimization Problems

1. The objective function is to be minimized
2. All variables are non-negative
3. All other constraints must have the form

$$ax + by \geq C$$

Steps 1: Setting up the Dual Table

1. Write the table of data from original problem
2. Put objective function at bottom (without minus signs)
3. Reverse the rows and columns (transpose)
4. Change all inequalities to $\leq$
5. Use s_1, s_2, ... for standard variables
6. Use original variables as the slack variables

Example 2

$$\text{Minimize } C = 10x + 11y$$

$$\begin{aligned} 20x + 10y &\geq 300 \\ 15x + 15y &\geq 300 \\ 10x + 20y &\geq 250 \end{aligned}$$

$$x \geq 0, y \geq 0$$

PRIMAL

	x	y	c
s_1	20	10	300
s_2	15	15	300
s_3	10	20	250
	10	11	

DUAL

s_1	s_2	s_3	
20	15	10	10
10	15	20	11
300	300	250	

CREATE A NEW SYSTEM OF INEQUALITIES

$$20s_1 + 15s_2 + 10s_3 \le 10$$
$$10s_1 + 15s_2 + 20s_3 \le 11$$

MAXIMIZE $P = 300s_1 + 300s_2 + 250s_3$

INITIAL SIMPLEX TABLE

s_1	s_2	s_3	x	y	P	C
20	$\boxed{15}$	10	1	0	0	10
10	15	20	0	1	0	11
-300	-300	-250	0	0	1	0

$\frac{1}{15}R_1 \rightarrow$

s_1	s_2	s_3	x	y	P	C
4/3	$\boxed{1}$	2/3	1/15	0	0	2/3
10	15	20	0	1	0	11
-300	-300	-250	0	0	1	0

$R_2 - 15R_1$
$R_3 + 300R_1$

s_1	s_2	s_3	x	y	P	C
4/3	1	2/3	1/15	0	0	2/3
-10	0	$\boxed{10}$	-1	1	0	1
100	0	-50	20	0	1	200

$\frac{1}{10}R_2$

s_1	s_2	s_3	x	y	P	C
4/3	1	2/3	1/15	0	0	2/3
-1	0	$\boxed{1}$	-1/10	1/10	0	1/10
100	0	-50	20	0	1	200

$R_1 - \frac{2}{3}R_2$

$R_3 + 50R_2$

s_1	s_2	s_3	x	y	P	C
2	1	0	2/15	-1/15	0	3/5
-1	0	1	-1/10	1/10	0	1/10
50	0	0	15	5	1	205

HOW DO I READ THE ANSWERS

READ DOWN THE COLUMN

$x = 15,\ y = 5,\ C = 205$

Example 3

$$\text{Minimize } C = 40x + 30y + 11z$$

$$\begin{aligned} 2x + y + z &\geq 8 \\ x + y - z &\geq 6 \\ x \geq 0, y &\geq 0 \end{aligned}$$

(1) PRIMAL

	x	y	z	c
s_1	2	1	1	8
s_2	1	1	-1	6
	40	30	11	

DUAL

s_1	s_2	
2	1	40
1	1	30
1	-1	11
8	6	

(2) NEW SYSTEM

MAXIMIZE $8s_1 + 6s_2$

SUBJECT TO

$$\begin{aligned} 2s_1 + 1s_2 &\leq 40 \\ 1s_1 + 1s_2 &\leq 30 \\ 1s_1 - s_2 &\leq 11 \end{aligned}$$

(3) SLACK VARIABLES

$$\begin{aligned} 2s_1 + s_2 + x &= 40 \\ 1s_1 + 1s_2 + y &= 30 \\ s_1 - s_2 + z &= 11 \end{aligned}$$

(4) INITIAL TABLE

s_1	s_2	x	y	z	p	c
2	1	1	0	0	0	40
1	1	0	1	0	0	30
$\boxed{1}$	-1	0	0	1	0	11
-8	-6	0	0	0	1	[illegible]

$R_1 - 2R_3$
$R_2 - R_3$
$R_4 + 8R_3$

s_1	s_2	x	y	z	p	c
0	3	1	0	-2	0	18
0	2	0	1	-1	0	19
1	-1	0	0	1	0	11
0	-14	0	0	8	1	88

WHAT'S THE NEXT PIVOT? 3

AFTER OPERATIONS

s_1	s_2	x	y	z	p	c
0	1	1/3	0	-2/3	0	6
0	0	-2/3	1	1/3	0	7
ϕ	0	1/3	0	1/3	0	17
0	0	14/3	0	-4/3	1	172

$3R_2$

s_1	s_2	x	y	z	p	c
0	1	1/3	0	-2/3	0	6
0	0	-2	3	1	0	21
ϕ	0	1/3	0	1/3	0	17
0	0	14/3	0	-4/3	1	172

	s_1	s_2	x	y	z	P	C
$R_1 + \frac{2}{3}R_2$	0	1	-1	2	0	0	20
$R_3 - \frac{1}{3}R_2$	0	0	-2	3	1	0	21
$R_4 + \frac{4}{3}R_2$	0	0	1	-1	0	0	10
	0	0	2	4	0	1	200

ANSWER: MINIMUM VALUE OF $C = 200$

WHEN $x = 2$, $y = 4$, $z = 0$

Example 4

An oil company operates two refineries. Refinery I has an ouput of 200, 100, and 100 barrels of low, medium, and high grade oil per day. Refinery II has an output of 100, 200, and 600 barrels of low, medium, and high grade oil per day. The company wishes to produce at least 1000, 1500, and 3000 of low, medium, and high grade oil to fill an order. If it costs \$200/day to operate Refinery I and \$300/day to operate Refinery II, determine how many days each refeinery should be operated to meet the production requirements at minimum cost to the company? What is the minimum cost?

X = # OF REFINERY I DAYS

y = # OF REFINERY II DAYS

LOW $200x + 100y \geq 1000$

MEDIUM $100x + 200y \geq 1500$

HIGH $100x + 600y \geq 3000$

$x \geq 0, y \geq 0$

OBJECTIVE IS MINIMIZE $C = \$200x + \$300y$

SINCE IT'S A STANDARD MIN, SET UP DUAL

PRIMAL TABLE

	X	y	CONST
	200	100	1000
	100	200	1500
	100	600	3000
OBJ →	200	300	

DUAL TABLE

S_1	S_2	S_3	CONST
200	100	100	200
100	200	600	300
1000	1500	3000	

NEW SYSTEM

$$200S_1 + 100S_2 + 100S_3 \le 200$$
$$100S_1 + 200S_2 + 600S_3 \le 300$$

MAXIMIZE $P = 1000S_1 + 1500S_2 + 3000S_3$

SLACK VARIABLES x, y

$$200S_1 + 100S_2 + 100S_3 + x = 200$$
$$100S_1 + 200S_2 + 600S_3 + y = 300$$

OBJ: $-1000S_1 - 1500S_2 - 3000S_3 + P = 0$

INITIAL TABLE

S_1	S_2	S_3	x	y	P	CONST
200	100	100	1	0	0	200
100	200	600	0	1	0	300
-1000	-1500	-3000	0	0	1	0

FINAL TABLE

S_1	S_2	S_3	x	y	P	CONST
1	0	$-4/3$	$1/150$	$-1/300$	0	$1/3$
0	0	$11/3$	$-1/30$	$1/150$	0	$4/3$
0	0	800	2	6	1	2200

ANSWER: $x = 2$,
$y = 6$

MINIMUM COST OF $C = \$2200$