MATH 210 FINITE MATHEMATICS

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2.6 Inverse of a Square Matrix

Definition 1: Inverse of Square Matrix

1. MATRIX MUST BE SQUARE $n \times n$
2. $A \cdot A^{-1} = I_n \qquad A^{-1} \cdot A = I_n$
3. MATRIX WITH NO INVERSE IS
4. CALLED SINGULAR

Example 1

Show that the following two matrices are inverses.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1(-2)+2(3/2) & 1(1)+2(-1/2) \\ 3(-2)+4(3/2) & 3(1)+4(-1/2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex] $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

FIND A^{-1}

START WITH $[A \mid I]$

$$= \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right]$$

NOW ROW REDUCE

$$\xrightarrow{R_2 + 1 R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array}\right]$$

$$\tfrac{1}{5}R_2 \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/5 & 1/5 \end{array}\right]$$

$$R_1 - 2R_2 \left[\begin{array}{cc|cc} 1 & 0 & 3/5 & -2/5 \\ 0 & 1 & 1/5 & 1/5 \end{array}\right]$$

FINAL: $A^{-1} = \begin{bmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{bmatrix}$

2x2 MATRIX SHORTCUT

GIVEN $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, THE INVERSE IS

$$A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

WHERE $D = ad - bc$

Ex $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

SO $$A^{-1} = \frac{1}{1(3) - 2(-1)}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{bmatrix}$$

NOTE: FOR A 2x2 MATRIX, IF [redacted] $ad - bc = 0$, THEN THE MATRIX HAS NO INVERSE.

Steps 1: Finding the Inverse of a Matrix

1. Adjoin the $n \times n$ identity matrix I to obtain the augmented matrix

$$[A \mid I]$$ ← START HERE

2. Use a sequence of row operations to reduce $[A|I]$ to

$$[I \mid B]$$

3. Matrix B is the inverse ⟶ So $B = A^{-1}$

4. If you don't end up with the identity matrix, then the inverse doesn't exist.

Example 2

Find the inverse to

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -3 \\ 2 & 1 & 1 \end{bmatrix}$$

START

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 3 & -3 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 + 1R_1$ ⟶ $R_3 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -2 & 1 & 1 & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \end{array}\right]$$

$\frac{1}{2}R_2$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \end{array}\right]$$

Continued

$$\xrightarrow[R_3 - 3R_2]{R_1 + 1R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 1/2 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 2 & -7/2 & -3/2 & 1 \end{array}\right]$$

$$\frac{1}{2}R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 1/2 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -7/4 & -3/4 & 1/2 \end{array}\right]$$

ROW 2 COLUMN 3 OF A^{-1}

$$R_2 + 1R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 1/2 & 0 \\ 0 & 1 & 0 & -5/4 & -1/4 & 1/2 \\ 0 & 0 & 1 & -7/4 & -3/4 & 1/2 \end{array}\right]$$

IS $1/2$

Example 3

Does $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 5 \end{bmatrix}$ have an inverse?

AFTER REDUCING $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$

WE GET $\left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 1 & 4/3 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{array}\right]$

- ROW OF 0S
- CAN'T HAVE AN INVERSE

Definition 2: Solving Systems of Equations with Inverses

Consider the following system:

$$\begin{aligned} 2x + y + z &= 1 \\ 3x + 2y + z &= 2 \\ 2x + y + 2z &= -1 \end{aligned}$$

B

1. Covert to the form $AX = B$ $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

2. Solution is $X = A^{-1}B$

1. Set up matrices

$$X = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

2. Find A^{-1}

SET UP $[A|I]$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

THEN ROW REDUCE

$$\tfrac{1}{2}R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1/2 & -3/2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$2R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$R_1 - \tfrac{1}{2}R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & \textcircled{1} & -1 & 0 & 1 \end{array}\right]$$

3. Solve for X

$$\begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

ANSWER: $X = A^{-1}B$

$$X = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} 3(1) - 1(2) - 1(-1) \\ -4(1) + 2(2) + 1(-1) \\ -1(1) + 0(2) + 1(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

SOLUTION FOR SYSTEM IS

$x = 2$
$y = -1$
$z = -2$

2.2 TECHNIQUE IS TO ROW REDUCE

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \\ 2 & 1 & 2 & -1 \end{array}\right]$$

MAGIC

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

But what if we have a new system?

$$\begin{aligned} 2x + y + z &= 2 \\ 3x + 2y + z &= -3 \\ 2x + y + 2z &= 1 \end{aligned}$$

BUT I CAN USE $X = A^{-1}B$

$$X = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -13 \\ -1 \end{bmatrix}$$

$x = 8$
$y = -13$
$z = -1$