MATH 210 FINITE MATHEMATICS

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2.2 System of Linear Equations - Unique Solutions

System of Equations **Augmented Matrix**

$$\begin{aligned} 3x - 2y &= 4 \\ 2x + 4y &= 8 \end{aligned}$$

$$\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 4 & 8 \end{array}\right]$$

$$\begin{aligned} 2x + 4y + 6z &= 22 \\ 2x + 8y + 5z &= 27 \\ -1x - 1y + 2z &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 2 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ -x + y + 2z &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$\begin{aligned} x + 0y + 0z &= 17 \rightarrow x = 17 \\ 0x + 1y + 0z &= -3 \rightarrow y = -3 \\ 0x + 0y + 1z &= 13 \rightarrow z = 13 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 13 \end{array}\right]$$

Definition 1: Row-Reduced Form of a Matrix

1. Each row consisting entirely of zeros must lie below rows having non-zero entries
2. The first non-zero entry in each row must be 1 (called a leading 1)
3. In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading one in the upper row.
4. If a column contains a leading 1, then the other entries in that column must be zeros

Example 1

Determine which of the following matrices are in row-reduced form.

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 13 \end{array}\right]$ YES

2. $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right]$ NO FAILS #3

3. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 2 & 7 \end{array}\right]$ NO FAILS #2

4. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$ YES

Definition 2: Gauss-Jordan Method

OPERATIONS

(1) SWAP ROWS IN THE MATRIX $R_1 \leftrightarrow R_2$

(2) MULTIPLY ANY ROW BY A NONZERO NUMBER

$$cR_1 \rightarrow R_1$$

(3) REPLACE A ROW BY THE SUM OF THAT ROW AND A CONSTANT MULTIPLE OF ANOTHER ROW

EX. $R_1 + 3R_2 \rightarrow R_1$

GOAL: TURN AUGMENTED MATRIX INTO

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

SOLUTION: $x=a,\ y=b,\ z=c$

Definition 3: Unit Column

A COLUMN WITH A SINGLE 1 AND THE REST ARE 0s

Example 2

Solve the following system using Gauss Jordan Method

$$\begin{aligned} 3x + 5y &= 9 \\ 2x + 3y &= 5 \end{aligned}$$

(1) WRITE AS AN AUGMENTED MATRIX

$$\left[\begin{array}{cc|c} 3 & 5 & 9 \\ 2 & 3 & 5 \end{array}\right]$$

(2) PIVOT $\frac{1}{3}R_1 \to R_1$ $\left[\begin{array}{cc|c} 1 & 5/3 & 3 \\ 2 & 3 & 5 \end{array}\right]$

$R_2 - 2R_1 \to R_2$

CHANGE PIVOT ROW

$(2) - 2(1) \to 0$
$(3) - 2(5/3) \to -1/3$
$(5) - 2(3) \to -1$

$$\left[\begin{array}{cc|c} 1 & 5/3 & 3 \\ 0 & -1/3 & -1 \end{array}\right]$$

(3) PIVOT $-3R_2 \to R_2$ $\left[\begin{array}{ccc} 1 & 5/3 & 3 \\ 0 & 1 & 3 \end{array}\right]$

$\underline{R_1 - \frac{5}{3}R_2}$
$(1) - \frac{5}{3}(0) \to 1$
$(\frac{5}{3}) - \frac{5}{3}(1) \to 0$
$(3) - \frac{5}{3}(3) \to -2$

$R_1 - \frac{5}{3}R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right]$$ DONE

$x = -2, \ y = 3$

Example 3

Solve

$$\begin{aligned} 2y + 3z &= 7 \\ 3x + 6y - 12z &= -3 \\ 5x - 2y + 2z &= -7 \end{aligned}$$

(1) WRITE IN AUGMENTED FORM

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 7 \\ 3 & 6 & -12 & -3 \\ 5 & -2 & 2 & -7 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|c} 3 & 6 & -12 & -3 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right]$$

$$\tfrac{1}{3}R_1 \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 5 & -2 & 2 & -7 \end{array}\right] \quad R_3 - 5R_1 \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 2 & 3 & 7 \\ 0 & -12 & 22 & -2 \end{array}\right]$$

$$\tfrac{1}{2}R_2 \left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 1 & 3/2 & 7/2 \\ 0 & -12 & 22 & -2 \end{array}\right] \quad \xrightarrow[R_3 + 12R_2]{R_1 - 2R_2} \quad \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 1 & 3/2 & 7/2 \\ 0 & 0 & 40 & 40 \end{array}\right]$$

$$\tfrac{1}{40}R_3 \left[\begin{array}{ccc|c} 1 & 0 & -7 & -8 \\ 0 & 1 & 3/2 & 7/2 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{c} R_1 + 7R_3 \\ R_2 - \tfrac{3}{2}R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x = -1, \quad y = 2, \quad z = 1$

Example 4

Brian wants to buy a 50 items from Amazon (board games and DVDs). The board games cost \$35 and the DVDs cost \$20. With a maximum of \$1600, how many board games and DVDs can he buy? Set up only.

LET $x = \#$ OF BOARD GAMES

$y = \#$ OF DVDS

RESTRICTIONS (CONSTRAINTS)

(1) $x + y = 50$

(2) $35x + 20y = 1600$

SET UP MATRIX

$$\begin{array}{l} \\ \text{PIVOT} \end{array} \left[\begin{array}{cc|c} 1 & 1 & 50 \\ 35 & 20 & 1600 \end{array}\right]$$

ROW REDUCE

$\xrightarrow{R_2 - 35R_1} \left[\begin{array}{cc|c} 1 & 1 & 50 \\ 0 & -15 & -150 \end{array}\right]$

$\underline{R_2 - 35R_1}$

$35 - 35(1) = 0$

$20 - 35(1) = -15$

$1600 - 35(50) = -150$

$-\frac{1}{15}R_2 \rightarrow \begin{bmatrix} 1 & 1 & 50 \\ 0 & 1 & 10 \end{bmatrix}$

$R_1 - 1R_2 \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 40 \\ 0 & 1 & 10 \end{array}\right]$

$x = 40$ GAMES, $y = 10$ DVDS