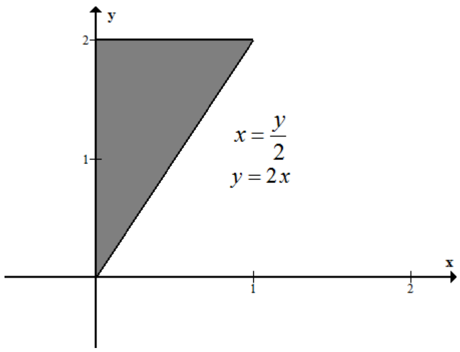
1. Double Integrals over General Regions

   (a) $\displaystyle\iint_D (x+y)\,dA$ where D is the triangle with vertices (0,0), (0,2), and (1,2).



Vertically Simple

$$\int_0^1 \int_{2x}^2 x+y\;dy\;dx$$

Inside

$$\begin{aligned}\int_{2x}^2 x+y\;dy &= xy+\frac{1}{2}y^2\Big|_{2x}^{2}\\ &= -4x^2+2x+2\end{aligned}$$

Outside

$$\begin{aligned}\int_0^1 -4x^2+2x+2\;dx &= -\frac{4}{3}x^3+x^2+2x\Big|_0^1\\ &= 5/3\end{aligned}$$

Horizontally Simple

$$\int_0^2 \int_0^{y/2} x+y\;dx\;dy$$

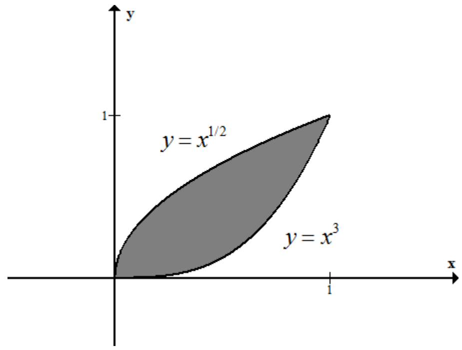
Inside

$$\begin{aligned}\int_0^{y/2} x+y\;dx &= \frac{1}{2}x^2+xy\Big|_0^{y/2}\\ &= \frac{5y^2}{8}\end{aligned}$$

Outside

$$\begin{aligned}\int_0^2 \frac{5}{8}y^2\;dy &= \frac{5}{24}y^3\Big|_0^2\\ &= 5/3\end{aligned}$$

(b) $\int\int_D 48xy\, dA$ where D is the region bounded by $y = x^3$ and $y = \sqrt{x}$.



Vertically Simple

$$\int_0^1 \int_{x^3}^{x^{1/2}} 48xy\; dy\; dx$$

Inside:

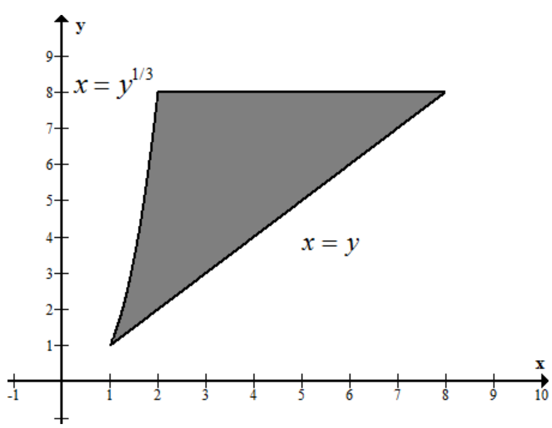
$$\begin{aligned}\int_{x^3}^{x^{1/2}} 48xy\; dy &= 24xy^2\Big|_{x^3}^{x^{1/2}} \\ &= 24x^2 - 24x^7\end{aligned}$$

Outside:

$$\begin{aligned}\int_0^1 24x^2 - 24x^7\; dx &= 8x^3 - 3x^8\Big|_0^1 \\ &= (8-3) - (0-0) \\ &= 5\end{aligned}$$

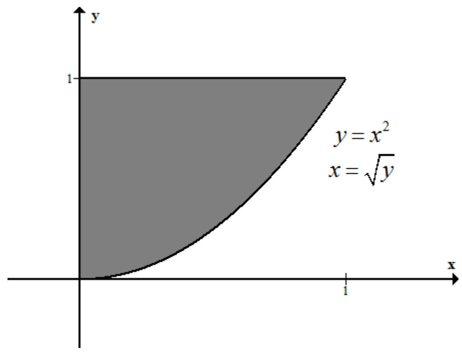
2. Reverse Order of Integration

(a) $\int_1^2 \int_x^{x^3} f(x,y)\; dy\; dx \;+\; \int_2^8 \int_x^8 f(x,y)\; dy\; dx$



$$\int_1^8 \int_{y^{1/3}}^{y} f(x,y)\; dx\; dy$$

(b) $\int_0^1 \int_{x^2}^1 \sqrt{y}\sin(y)\ dy\ dx$



$$\int_0^1 \int_{x^2}^1 \sqrt{y}\sin(y)\ dy\ dy = \int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\sin(y)\ dx\ dy$$

Inside:

$$\begin{aligned}\int_0^{\sqrt{y}} \sqrt{y}\sin(y)\ dx &= \sqrt{y}\sin(y)x|_0^{\sqrt{y}} \\ &= y\sin(y)\end{aligned}$$

Outside:

$$\int_0^1 y\sin(y)\ dy$$

i. Let $u = y$, $dv = \sin(y)\ dy$
ii. $du = dy$, and $v = -\cos(y)$

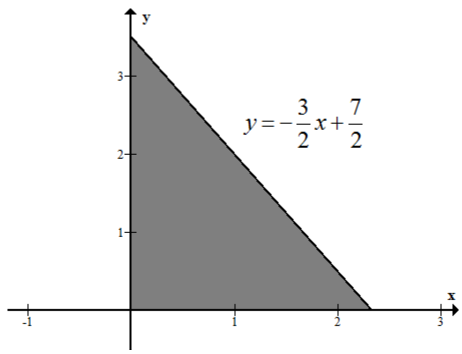
$$\begin{aligned}\int_0^1 y\sin(y)\ dy &= -y\cos(y) + \int \cos(y)\ dy \\ &= -y\cos(y) + \sin(y)\end{aligned}$$

$$-y\cos(y) + \sin(y)|_0^1 = (-\cos(1) + \sin(1)) - (0 + 0) = \sin(1) - \cos(1)$$

3. Find Volume of solid

(a) Tetrahedron in first octant bounded by coordinate planes $z = 7 - 3x - 2y$. Hint: To find D, let $z = 0$ to get the projection onto the xy plane.

If we let $z = 0$ then D is the region on the xy plane restricted to the first quadrant and the line $3x + 2y = 7$.



$$\text{Volume } = \int\int_D f(x,y)\, dA = \int_0^{7/3} \int_0^{-\frac{3}{2}x+\frac{7}{2}} 7 - 3x - 2y\; dy\; dx$$

Inside:

$$\begin{aligned}\int_0^{-\frac{3}{2}x+\frac{7}{2}} 7 - 3x - 2y\; dy &= 7y - 3xy - y^2\Big|_0^{-\frac{3}{2}x+\frac{7}{2}} \\ &= 7\left(-\frac{3}{2}x + \frac{7}{2}\right) - 3x\left(-\frac{3}{2}x + \frac{7}{2}\right) - \left(-\frac{3}{2}x + \frac{7}{2}\right)^2 \\ &= \frac{9}{4}x^2 - \frac{21}{2}x + \frac{49}{4}\end{aligned}$$

Outside:

$$\begin{aligned}\int_0^{7/3} \frac{9}{4}x^2 - \frac{21}{2}x + \frac{49}{4}\; dx &= \frac{3}{4}x^3 - \frac{21}{4}x^2 + \frac{49}{4}x\Big|_0^{7/3} \\ &= 343/36\end{aligned}$$

4. Polar Double Integrals

(a) $\int_0^{\pi/2} \int_1^3 re^{-r^2}\, dr\, d\theta$

Inside:

$$\begin{aligned} \int_1^3 re^{-r^2}\, dr\, d\theta &= \int_{r=1}^{r=3} -\frac{1}{2}e^u\, du \\ &= -\frac{1}{2}e^u\Big|_{r=1}^{r=3} \\ &= -\frac{1}{2}e^{-r^2}\Big|_1^3 \\ &= -\frac{1}{2}e^{-9} + \frac{1}{2}e^{-1} \end{aligned}$$

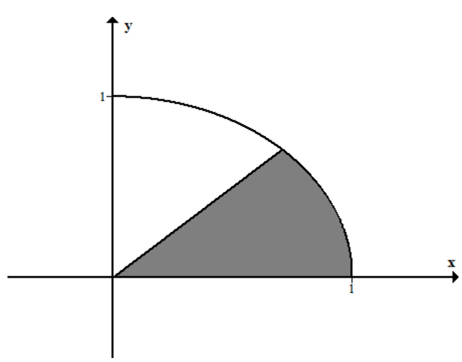
Outside:

$$\begin{aligned} \int_0^{\pi/2} -\frac{1}{2}e^{-9} + \frac{1}{2}e^{-1}\, d\theta &= \left(-\frac{1}{2}e^{-9} + \frac{1}{2}e^{-1}\right)\theta\Big|_0^{\pi/2} \\ &= \left(-\frac{1}{2}e^{-9} + \frac{1}{2}e^{-1}\right)\cdot\frac{\pi}{2} \end{aligned}$$

Answer: $-\frac{1}{4}e^{-9}\pi + \frac{1}{4}e^{-1}\pi$

5. Convert from Cartesian to Polar

(a) Find $\int\int_D xy\, dA$ where D is the region bounded by the x-axis, the line $y = x$, and the circle $x^2 + y^2 = 1$



$$\begin{aligned} \int\int_D xy\, dA &= \int_0^{\pi/4}\int_0^1 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta \\ &= \int_0^{\pi/4}\int_0^1 r^3\cos\theta\sin\theta\, dr\, d\theta \\ &= \int_0^{\pi/4} \cos\theta\sin\theta\, d\theta \cdot \int_0^1 r^3\, dr \end{aligned}$$

Left side:

$$\int_0^{\pi/4} \cos\theta \sin\theta \, d\theta$$

$$\begin{aligned}
& u = \sin\theta, \quad du = \cos\theta \, d\theta \\
&= \int_{\theta=0}^{\theta=\pi/4} u \, du \\
&= \frac{1}{2}u^2 \Big|_{\theta=0}^{\theta=\pi/4} \\
&= \frac{1}{2}\sin^2(\theta)\Big|_0^{\pi/4} \\
&= \frac{1}{2}\sin^2(\pi/4) - \frac{1}{2}\sin^2(0) \\
&= 1/4
\end{aligned}$$

Right Side:

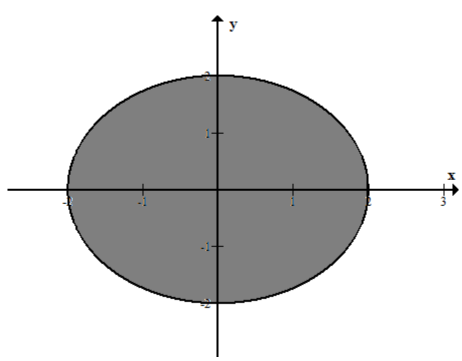
$$\begin{aligned}
\int_0^1 r^3 \, dr &= \frac{1}{4}r^4 \Big| 0^1 \\
&= 1/4
\end{aligned}$$

Final:

$$\int_0^{\pi/4} \cos\theta \sin\theta \, d\theta \cdot \int_0^1 r^3 \, dr = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$$

(b) Find the volume of the solid bounded by $z = 4 - x^2 - y^2$ and the xy-plane. Hint: to find D, set $z = 0$ to get the projection onto the xy plane.

If we let $z = 0$ then D is the region on the xy plane restricted to $0 = 4 - x^2 - y^2$. This is equivalent to $x^2 + y^2 = 4$.



$$\text{Volume } = \int\int_D f(x,y) \, dA = \int\int_D 4 - x^2 - y^2 \, dA$$

We can either describe D (the shaded region) in rectangular coordinates or polar. As D is a circular it is best that we use polar.

$$D = \{(r,\theta) \mid 0 \le r \le 2, \; 0 \le \theta \le 2\pi\}$$

We need to convert our function $f(x,y) = 4 - x^2 - y^2$ into a polar.

$$4 - x^2 - y^2 = 4 - (x^2 + y^2) = 4 - r^2$$

Set up the integral

$$V = \int_0^{2\pi} \int_0^2 (4 - r^2)\ r\ dr\ d\theta = \int_0^{2\pi} \int_0^2 4r - r^3\ dr\ d\theta$$

Inside:

$$\begin{aligned}
\int_0^2 4r - r^3\ dr &= 2r^2 - \frac{1}{4}r^4 \Big|_0^2 \\
&= \left[2(2)^2 - \frac{1}{4}(2)^4\right] - \left[2(0)^2 - \frac{1}{4}(0)^4\right] \\
&= 4
\end{aligned}$$

Outside:

$$\begin{aligned}
\int_0^{2\pi} 4\ d\theta &= 4\theta|_0^{2\pi} \\
&= 4(2\pi) - 4(0) \\
&= 8\pi
\end{aligned}$$