Math 232 Final (Fall 2010) Section (   )

NAME (Print): SOLUTION

ZID :

1. (20 pts) Let P(1, 2, 3) and Q(1, −1, −2) be two points in the three dimensional space.

(a) Find the area of the triangle with vertices at P, Q, and the origin. LET $R=(0,0,0)$

- $\vec{PR} = \langle 0-1, 0-2, 0-3 \rangle = \langle -1, -2, -3 \rangle$

  $\vec{PQ} = \langle 0, -3, -5 \rangle$

- AREA OF TRIANGLE IS $\frac{1}{2} | \vec{PQ} \times \vec{PR} |$

- $\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 0 & -3 & -5 \\ -1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} -3 & -5 \\ -2 & -3 \end{vmatrix} i - \begin{vmatrix} 0 & -5 \\ -1 & -3 \end{vmatrix} j + \begin{vmatrix} 0 & -3 \\ -1 & -2 \end{vmatrix} k$

$$= (9-10)i - (0-5)j + (0-3)k$$

$$= -1i + 5j - 3k$$

- $|\vec{PQ} \times \vec{PR}| = \sqrt{(-1)^2 + (5)^2 + (-3)^2} = \sqrt{35}$

- AREA $= \frac{1}{2}\sqrt{35}$

(b) Find an equation of the line through P(1, 2, 3) and that is perpendicular (orthogonal) to the plane through P, Q, and the origin.

NORMAL VECTOR TO THE PLANE $\langle -1, 5, -3 \rangle$ CAN ACT AS THE DIRECTION VECTOR OF THE LINE

LINE: $x = -1t + 1$, $y = 5t + 2$, $z = -3t + 3$

2. (10 pts) Find an equation of the tangent plane to the surface $z = e^{xy}$ at the point P$(1, 1, e)$. FORMULA: $z - z_0 = f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)$

- $x_0 = 1,\ y_0 = 1,\ z_0 = e$
- $f_x = ye^{xy}$, $f_x(1,1) = 1e^1 = e$
- $f_y = xe^{xy}$, $f_y(1,1) = 1e^1 = e$
- $z - e = e(x-1) + e(y-1)$

$$z - e = ex - e + ey - e$$

$$z = ex + ey - e \quad \text{OR} \quad ex + ey - z = e$$

3. (10 pts) Suppose the equation $xy + e^{xyz} - z - e^y = 0$ implicitly defines each of the variables as functions of the other two variables. Use implicit partial derivative to find $\dfrac{\partial z}{\partial x}$ at the point P$(1, 1, 1)$. LET $F(x,y,z) = xy + e^{xyz} - z - e^y$

1ST WAY • $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z} = \dfrac{-(y + yze^{xyz})}{xye^{xyz} - 1}$ AND $\left.\dfrac{\partial z}{\partial x}\right|_{(1,1,1)} = \dfrac{-(1 + 1e^1)}{1e^1 - 1} = \dfrac{-(1+e)}{e-1}$

2ND WAY • $\dfrac{\partial}{\partial x}\left[xy + e^{xyz} - z - e^y\right] = \dfrac{\partial}{\partial x}[0]$

$$\Rightarrow y + e^{xyz} \cdot \left(yz + xy \cdot \frac{\partial z}{\partial x}\right) - 1\frac{\partial z}{\partial x} = 0$$

SAME

$$\Rightarrow y + yze^{xyz} + xye^{xyz}\frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} = 0$$

SOLVE FOR $\dfrac{\partial z}{\partial x}$ $\Rightarrow \dfrac{\partial z}{\partial x} = \dfrac{-y - yze^{xyz}}{xye^{xyz} - 1}$ AND $\left.\dfrac{\partial z}{\partial x}\right|_{(1,1,1)} = \dfrac{-1 - 1e^1}{1e^1 - 1} = \dfrac{-1-e}{e-1}$

4. (10 pts) Decide a scalar q so that two planes $qx + y + z = 3$ and $x - y + 5z = 0$ are perpendicular. MEANS THEIR NORMAL VECTORS ARE PERPENDICULAR

- $\vec{n}_1 = \langle q, 1, 1 \rangle$, $\vec{n}_2 = \langle 1, -1, 5 \rangle$
- SOLVE $\vec{n}_1 \cdot \vec{n}_2 = 0$

$$\langle q, 1, 1 \rangle \cdot \langle 1, -1, 5 \rangle = 0$$
$$q - 1 + 5 = 0$$
$$q = -4$$

5.(16 pts) Let $f(x, y) = x^2 - 5xy$.

a) Find the directional derivative of the function at (1,2) in the direction of $\vec{v} = 3\vec{i} + 4\vec{j}$.

- $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$
- $D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u} = \langle 2x - 5y, -5x \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$
- $D_{\vec{u}} f(1,2) = \langle 2\cdot1 - 5\cdot2, -5\cdot1 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$

$$= \langle -8, -5 \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle$$
$$= -8\left(\tfrac{3}{5}\right) + -5\left(\tfrac{4}{5}\right) = \frac{-44}{5}$$

b) Find the direction in which the function increases most rapidly at (1,2). Then find the derivative in that direction.

FUNCTION INCREASE MOST RAPIDLY IN THE DIRECTION $\nabla f(1,2)$

$$\nabla f(1,2) = \langle -8, -5 \rangle$$

DERIVATIVE IS $|\nabla f(1,2)| = |\langle -8, -5 \rangle|$

$$= \sqrt{64 + 25}$$
$$= \sqrt{89}$$

6. (14 pts) Find the local maxima, local minima, and saddle points of

$f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - xy + 4$, if there is any.

(1) $f_x = x^2 - y$, $f_y = y^2 - x$, $f_{xx} = 2x$, $f_{yy} = 2y$, $f_{xy} = -1$

(2) SOLVE $x^2 - y = 0$ AND $y^2 - x = 0$

$\downarrow$

$y = x^2$ PLUG INTO $y^2 - x = 0 \Rightarrow (x^2)^2 - x = 0$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0, 1$

(3) TWO CRITICAL POINTS $(0, 0)$ AND $(1, 1)$

(4) $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2x & -1 \\ -1 & 2y \end{vmatrix} = 4xy - 1$

$(0,0)$: $D = 4(0)(0) - 1 = -1 < 0$ SADDLE POINT

$(1,1)$: $D = 4(1)(1) - 1 = 3 > 0$

SINCE $f_{xx}(1,1) = 2(1) > 0$ AND $D > 0$, THEN

$(1,1)$ IS A LOCAL MINIMUM

7. (15 pts) Use the method of Lagrange multipliers to find the point on the plane $x+2y+3z=6$ that is closest to the origin. (Hint: minimize the square of the distance: $d^2 = x^2+y^2+z^2$ to make algebra simpler.)

LET $f(x,y,z) = x^2+y^2+z^2$ WHERE $g(x,y,z) = x+2y+3z$

(1) $\nabla f = \langle 2x, 2y, 2z \rangle$, $\nabla g = \langle 1, 2, 3 \rangle$

(2) SOLVE $\nabla f = \lambda \nabla g$ AND $g = 6$

$$\begin{cases} 2x = \lambda \cdot 1 \\ 2y = \lambda \cdot 2 \\ 2z = \lambda \cdot 3 \\ x+2y+3z = 6 \end{cases} \Rightarrow$$

SOLVE ~~2x~~ FOR x, y, z IN TERMS OF λ

$x = \frac{1}{2}\lambda$, $y = \lambda$, $z = \frac{3}{2}\lambda$

(3) PLUG $x = \frac{1}{2}\lambda$, $y = \lambda$, $z = \frac{3}{2}\lambda$ INTO $x+2y+3z=6$

$$\frac{1}{2}\lambda + 2(\lambda) + 3\left(\frac{3}{2}\lambda\right) = 6$$

$$\frac{14}{2}\lambda = 6$$

$$\lambda = \frac{12}{14} = \frac{6}{7}$$

(4) $x = \frac{1}{2} \cdot \left(\frac{6}{7}\right) = \frac{6}{14}$

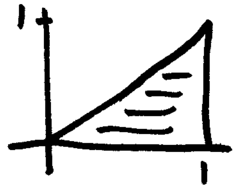
$y = \frac{6}{7}$

$z = \frac{3}{2}\left(\frac{6}{7}\right) = \frac{9}{7}$

$$\left(\frac{6}{14}, \frac{6}{7}, \frac{9}{7}\right)$$

8. (14 pts) Evaluate $\int_0^1 \int_y^1 \frac{e^x}{x}\, dx\, dy$ (Hint: reverse the order of integration.)

(1) $D = \begin{cases} 0 \le y \le 1 \\ y \le x \le 1 \end{cases}$



(2) CHANGE ORDER TO $D = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$ $\quad \int_0^1 \int_0^x \frac{e^x}{x}\, dy\, dx$

(3) $\int_0^x \frac{e^x}{x}\, dy = \frac{e^x}{x} y \Big|_{y=0}^{y=x} = \frac{e^x}{x} \cdot x - \frac{e^x}{x} \cdot 0 = e^x$

$\int_0^1 e^x\, dx = e^x \Big|_0^1 = e^1 - e^0 = \underline{\underline{e - 1}}$

9. (14 pts) Compute the double integral $\iint_D (y - 5x)^3 (2y + x)^4\, dA$ where D is the region bounded by $y = 5x$, $y = 5x + 1$, $2y = -x - 1$, and $2y = -x + 1$.

(Hint: Use the transformation $u = y - 5x$ and $v = 2y + x$.

NOTE:

(1)
$y = 5x \Rightarrow y - 5x = 0 \Rightarrow u = 0$
$y = 5x + 1 \Rightarrow y - 5x = 1 \Rightarrow u = 1$
$2y = -x + 1 \Rightarrow 2y + x = 1 \Rightarrow v = 1$
$2y = -x - 1 \Rightarrow 2y + x = -1 \Rightarrow v = -1$

(2) $\begin{matrix} u = y - 5x \\ v = 2y + x \end{matrix} \Rightarrow \begin{matrix} -2u = -2y + 10x \\ v = 2y + x \end{matrix}$
$\Rightarrow -2u + v = 11x$
$\Rightarrow x = \frac{-2u + v}{11}$

$\begin{matrix} u = y - 5x \\ v = 2y + x \end{matrix} \Rightarrow \begin{matrix} u = y - 5x \\ 5v = 10y + 5x \end{matrix}$
$\Rightarrow u + 5v = 11y$
$\Rightarrow y = \frac{u + 5v}{11}$

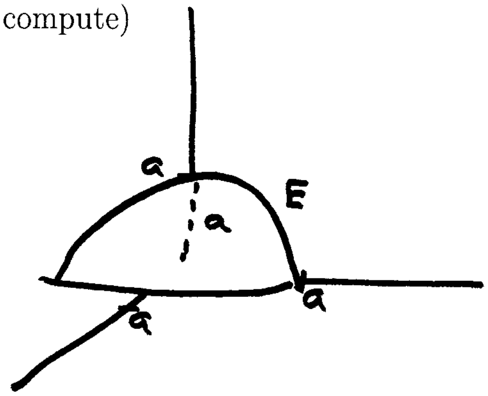
(3) $J = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{-2}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{5}{11} \end{vmatrix} = \frac{-10}{121} - \frac{1}{121} = \frac{-1}{11}$

(4) $\frac{-1}{11} \int_0^1 \int_{-1}^1 u^3 v^4\, dv\, du = -\frac{1}{11} \cdot \int_0^1 u^3\, du \cdot \int_{-1}^1 v^4\, dv$

$= -\frac{1}{11} \cdot \frac{1}{4} u^4 \Big|_0^1 \cdot \frac{1}{5} v^5 \Big|_{-1}^1 = -\frac{1}{11} \cdot \frac{1}{4} \cdot \frac{2}{5} = \frac{-1}{110}$

10. (20 pts) Set up the following triple integrals to find the volume of the upper hemi-sphere of radius a, $x^2 + y^2 + z^2 = a^2$ and $z \geq 0$ (Do not compute)

a) a triple integral in cylindrical coordinates :



$$E = \begin{cases} 0 \leq r \leq a \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq \sqrt{a^2 - x^2 - y^2} \end{cases}$$

$$\int_0^a \int_0^{2\pi} \int_0^{\sqrt{a^2-x^2-y^2}} 1 \cdot r \, dz \, d\theta \, dr$$

b) a triple integral in spherical coordinates :

$$E = \begin{cases} 0 \leq \rho \leq a \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/2 \end{cases}$$

$$\int_0^a \int_0^{2\pi} \int_0^{\pi/2} 1 \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

11. (32 pts) Let $\mathbf{F} = < xy^2 + 2y,\ x^2y + 2x + 2 >$ be a vector field in the two dimensional space.

a) Show that $\mathbf{F}$ is conservative.

$P(x,y) = xy^2 + 2y \qquad Q(x,y) = x^2y + 2x + 2$

$\frac{\partial P}{\partial y} = 2xy + 2 \ , \ \frac{\partial Q}{\partial x} = 2xy + 2$

CONSERVATIVE SINCE $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

b) Find a potential function f for the field $\mathbf{F}$.

$f = \int P(x,y)\,dx = \int xy^2 + 2y\,dx = \frac{1}{2}x^2y^2 + 2xy + g(y)$

AND $f_y = x^2y + 2x + g'(y)$. MATCHING $f_y = Q(x,y)$

WE GET $g'(y) = 2$, SO $g(y) = 2y + C$

POTENTIAL FUNCTION: $f = \frac{1}{2}x^2y^2 + 2xy + 2y + C$

c) Compute the line integral $\int_c \mathbf{F} \cdot d\mathbf{r}$ where c is a smooth curve $\mathbf{r}(t) = < e^t, 1 + t >,\ 0 \le t \le 1$.

SINCE F IS CONSERVATIVE, $F = \nabla f$

THEN $\int_c F \cdot dr = f(r(1)) - f(r(0))$ WHERE $r(1) = \langle e, 2 \rangle$
$r(0) = \langle 1, 1 \rangle$

$\Rightarrow f(e,2) - f(1,1) = \left[\frac{1}{2}e^2 \cdot 4 + 2 \cdot e \cdot 2 + 2 \cdot 2\right] - \left[\frac{1}{2} + 2 + 2\right]$

$= 2e^2 + 4e - \frac{1}{2}$

d) Compute the line integral $\int_c \mathbf{F} \cdot d\mathbf{r}$ where c is a closed curve $\mathbf{r}(t) = < 2\sin t, \cos t >,\ 0 \le t \le 2\pi$.

$\int_c F \cdot dr = 0$ SINCE C IS CLOSED OVER A CONSERVATIVE VECTOR FIELD

12.(10 pts) A wire of density $\delta(x,y) = x$ lies along the curve

$\mathbf{r}(t) = < t,\ t^2 >,\ 0 \le t \le 2$. Find the mass of the wire. $m = \int_C \delta(x,y)\,ds$

WHERE C IS $r(t) = < t, t^2 >$

$\uparrow$ $x(t)$ $\uparrow$ $y(t)$ AND $x'(t) = 1,\ y'(t) = 2t$

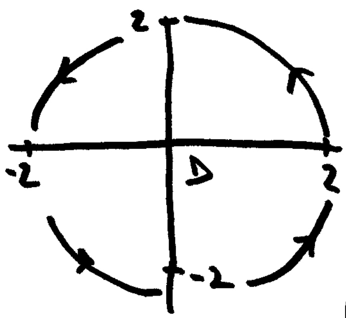
(1) $m = \int_C x\,ds = \int_0^2 t \cdot \sqrt{(1)^2 + (2t)^2}\,dt = \int_0^2 t\sqrt{1+4t^2}\,dt$

(2) LET $u = 1+4t^2$, $du = 8t\,dt$

$\Rightarrow \int_1^{17} \frac{1}{8} u^{1/2}\,du = \frac{1}{12} u^{3/2}\Big|_1^{17} = \frac{1}{12}(17)^{3/2} - \frac{1}{12}(1)^{3/2}$

13.(15 pts) Evaluate $\int_c -y^3\,dx + x^3\,dy$ where c is a circle given by

$\mathbf{r}(t) = < 2\cos t,\ 2\sin t >,\ 0 \le t \le 2\pi$. (Hint: Use Green's theorem)



$P(x,y) = -y^3,\ Q(x,y) = x^3$

$P_y = -3y^2,\ Q_x = 3x^2$

$\int P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA$

$= \iint_D 3x^2 + 3y^2\,dA$

$* D = \begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$

AND

$3x^2 + 3y^2 = 3r^2$

$= \int_0^{2\pi}\int_0^2 3r^2 \cdot r\,dr\,d\theta$

$= \int_0^{2\pi} 1\,d\theta \cdot \int_0^2 3r^3\,dr$

$= \theta\Big|_0^{2\pi} \cdot \frac{3r^4}{4}\Big|_0^2 = 2\pi \cdot 12 = 24\pi$