MATH 232 FINAL EXAM Name (print) SOLUTION

Fall 2015 ZID

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1. **(10 points)** Let $z = \ln(\frac{x}{y})$, where $x = u^2 + 2v^2$, and $y = uv$. Find $\frac{\partial z}{\partial u}$ when $u = 1$, and $v = 1$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

- $\frac{\partial z}{\partial x} = \frac{1}{x/y} \cdot \frac{1}{y} = \frac{1}{x}$ , $\frac{\partial z}{\partial y} = \frac{1}{x/y} \cdot \frac{-x}{y^2} = \frac{-1}{y}$
- $\frac{\partial x}{\partial u} = 2u$ $\qquad \frac{\partial y}{\partial u} = v$
- IF $u=1, v=1$ THEN $x = 1^2 + 2 \cdot 1^2 = 3$ , $y = 1 \cdot 1 = 1$
- $\left.\frac{\partial z}{\partial x}\right|_{(1,1)} = \frac{1}{3}$ , $\left.\frac{\partial z}{\partial y}\right|_{(1,1)} = \frac{-1}{1} = -1$
- $\left.\frac{\partial x}{\partial u}\right|_{(1,1)} = 2 \cdot 1 = 2$ , $\left.\frac{\partial y}{\partial u}\right|_{(1,1)} = 1$
- FINAL: $\frac{\partial z}{\partial u} = \frac{1}{3} \cdot 2 + -1 \cdot 1 = -1/3$

2. **(15 points)** Consider the curve $C$ given by $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + 3t\mathbf{k}$, $0 \le t \le \pi$.

(a) Find the derivative of $\mathbf{r}(t)$.

$$r'(t) = -2\sin(2t)i + 2\cos(2t)j + 3k$$

(b) Find parametric equations of the tangent line to the curve at $t = \pi/4$.

WANT: $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$

(1) $\langle x_0, y_0, z_0 \rangle = r(\pi/4) = \langle \cos(\pi/2), \sin(\pi/2), 3 \cdot \pi/4 \rangle = \langle 0, 1, 3\pi/4 \rangle$

(2) $\langle a, b, c \rangle = r'(\pi/4) = \langle -2\sin(\pi/2), 2\cos(\pi/2), 3 \rangle = \langle -2, 0, 3 \rangle$

(3) FINAL:

$$x = 0 + -2t, \quad y = 1 + 0t, \quad z = \frac{3\pi}{4} + 3t$$

$$\Rightarrow \boxed{x = -2t, \quad y = 1, \quad z = \frac{3\pi}{4} + 3t}$$

(c) Find the length of the curve.

IF $r(t) = \langle x(t), y(t), z(t) \rangle = \langle \cos 2t, \sin 2t, 3t \rangle$

THEN LENGTH IS $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$$\int_0^\pi \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2 + (3)^2}\, dt = \int_0^\pi \sqrt{4\sin^2 2t + 4\cos^2 2t + 9}\, dt$$

$$= \int_0^\pi \sqrt{13}\, dt = \sqrt{13}\, t \Big|_0^\pi = \sqrt{13}\,\pi$$

3. **(15 points)** For each of the following, either show that the limit exists and find the limit, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y-x}{\sqrt{x^2+y^2}}$

ALONG $y=0$: $\displaystyle\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0-x}{\sqrt{x^2+0^2}} = \lim_{x\to 0} \frac{-x}{\sqrt{x^2}} = \begin{cases} \frac{-x}{-x}=1, & x<0 \\ \frac{-x}{x}=-1, & x>0 \end{cases}$

$\sqrt{x^2}=|x|$

WE GET TWO DIFFERENT LIMIT VALUES WHEN APPROACHING THE ORIGIN, SO $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y-x}{\sqrt{x^2+y^2}}$ DNE

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}}$ CHANGE TO POLAR

* $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2$

$$\lim_{r\to 0} \frac{2r\cos\theta\, r\sin\theta}{\sqrt{r^2}} = \lim_{r\to 0} \frac{2r^2\cos\theta\sin\theta}{r} = \lim_{r\to 0} 2r\cos\theta\sin\theta$$
$$= 0$$

THE LIMIT EXISTS AND IT EQUALS 0

4. **(15 points)** Let $P(1,0,-1), Q(1,2,3)$ and $O(0,0,0)$ be three points in $\mathbb{R}^3$.

(a) Find $\vec{OP} \times \vec{OQ}$.

$$\vec{OP} = \langle 1-0,\ 0-0, -1-0\rangle = \langle 1, 0, -1\rangle$$
$$\vec{OQ} = \langle 1-0,\ 2-0, 3-0\rangle = \langle 1, 2, 3\rangle$$

$$\vec{OP} \times \vec{OQ} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 2 & 3 \end{vmatrix} = i\begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$
$$= 2i - j(+4) + 2k$$
$$= 2i - 4j + 2k$$
$$= \langle 2, -4, 2\rangle$$

(b) Find an equation of the plane passing through the points $P, Q$, and $O$.

WE NEED NORMAL VECTOR AND POINT

EASIEST ↓

NORMAL VECTOR: $\langle a, b, c\rangle = \langle 2, -4, 2\rangle$, $\langle x_0, y_0, z_0\rangle = \langle 0, 0, 0\rangle$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$
$$2(x-0) + -4(y-0) + 2(z-0) = 0$$
$$2x - 4y + 2z = 0 \quad \text{OR} \quad x - 2y + z = 0$$

(c) Find the area of triangle $POQ$.

$$\text{AREA} = \frac{1}{2}\left|\vec{OP} \times \vec{OQ}\right|$$
$$= \frac{1}{2}\left|\langle 2, -4, 2\rangle\right|$$
$$= \frac{1}{2}\sqrt{(2)^2 + (-4)^2 + (2)^2}$$
$$= \frac{1}{2}\sqrt{24} = \sqrt{6}$$

5. **(15 points)** Let $f(x, y) = \sqrt{x^2 + 2y^2 + 1}$.

(a) Find $\nabla f(x, y)$.

$$\nabla f(x,y) = \left\langle \frac{2x}{2\sqrt{x^2+2y^2+1}}, \frac{4y}{2\sqrt{x^2+2y^2+1}} \right\rangle$$

$$= \left\langle \frac{x}{\sqrt{x^2+2y^2+1}}, \frac{2y}{\sqrt{x^2+2y^2+1}} \right\rangle$$

(b) Find the linearization $L(x, y)$ of $f$ at $(1, 1)$.

$$L(x,y) = z_0 + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$= 2 + \frac{1}{2}(x-1) + \frac{2}{2}(y-1)$$

$$= 2 + \frac{1}{2}x - \frac{1}{2} + y - 1$$

$$= \frac{x}{2} + y + \frac{1}{2}$$

(c) Use the linearization in (b) to approximate $f(0.9, 1.1)$.

$$f(.9, 1.1) \approx L(.9, 1.1) = \frac{.9}{2} + 1.1 + \frac{1}{2}$$

$$= 2.05$$

6. **(16 points)** Let $f(x, y) = x^2y - x^2 - 2y^2$.

(a) Find all critical points of $f$.

$f_x = 2xy - 2x$ , $f_y = x^2 - 4y$

SOLVE $f_x = 2xy - 2x = 0$
AND $f_y = x^2 - 4y = 0$

(1) $2xy - 2x = 0$
$\Rightarrow 2x(y-1) = 0$
$\Rightarrow x = 0$ OR $y = 1$

USE $x^2 - 4y = 0$
TO FIND THE
OTHER COORDINATE

(2) $x = 0$: $x^2 - 4y = 0$
$0^2 - 4y = 0$ $\Rightarrow (0,0)$
$y = 0$

$y = 1$: $x^2 - 4y = 0$
$x^2 - 4 = 0$ $\Rightarrow$ $(-2, 1)$
$x = -2, 2$ $\quad (2, 1)$

3 CRITICAL POINTS $(0,0), (-2,1), (2,1)$

(b) Use the Second Derivatives Test to find the local maxima, local minima, and saddle points of $f$ if there are any.

• $f_{xx} = 2y - 2$ , $f_{yy} = -4$ , $f_{xy} = 2x$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2y-2 & 2x \\ 2x & -4 \end{vmatrix} = -4(2y-2) - (2x)^2$$
$$= -8y + 8 - 4x^2$$

$(0,0)$: $D = -8(0) + 8 - 4(0)^2 = 8 > 0$ , $f_{xx}(0,0) = 2(0) - 2 = -2$
SO $(0,0)$ IS A LOCAL MAX

$(-2,1)$: $D = -8(1) + 8 - 4(-2)^2 = -16 < 0 \rightarrow$ SADDLE POINT

$(2,1)$: $D = -8(1) + 8 - 4(2)^2 = -16 < 0 \rightarrow$ SADDLE POINT

ANSWER:
MIN OCCURS AT $\left(-\sqrt{4/3}, \pm\sqrt{4/3}\right) \to f\left(-\sqrt{4/3}, \pm\sqrt{4/3}\right) = \dfrac{-8}{3\sqrt{3}}$

7. **(12 points)** Use the method of Lagrange multipliers to find the minimum value of $f(x, y) = xy^2$ subject to the constraint $x^2 + 2y^2 = 4$. LET $g(x,y) = x^2 + 2y^2$

(1) SOLVE $\begin{cases} \nabla f = \lambda \nabla g \\ g = 4 \end{cases} \Rightarrow \begin{cases} y^2 = \lambda \cdot 2x \\ 2xy = \lambda \cdot 4y \\ x^2 + 2y^2 = 4 \end{cases}$

(2) $2xy = \lambda 4y$

$\Rightarrow 2xy - 4\lambda y = 0$

$\Rightarrow 2y(x - 2\lambda) = 0$

SO $y = 0$ OR $x = 2\lambda$

$\Rightarrow \lambda = \dfrac{x}{2}$

(3) LET $y = 0$:

$y^2 = 2\lambda x$

$0 = 2\lambda x$

SO $\lambda = 0$ OR $\cancel{x = 0}$ CAN'T

PLUG $y = 0$ INTO $x^2 + 2y^2 = 4$

$x^2 = 4$

$x = \pm 2$

(4) LET $\lambda = x/2$

$y^2 = 2\lambda x$

$y^2 = 2 \cdot \frac{x}{2} x$

$y^2 = x^2$

PLUG $y^2 = x^2$ INTO $x^2 + 2y^2 = 4$

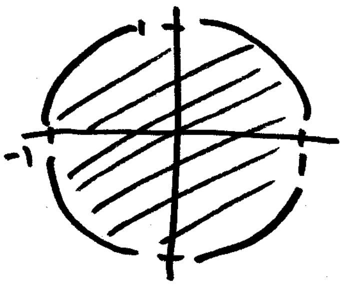
$y^2 + 2y^2 = 4$

$3y^2 = 4 \Rightarrow y = \pm\sqrt{4/3}$

CHECK FOR SMALLEST $(-2, 0)$, $(2, 0)$, $\left(\sqrt{4/3}, \sqrt{4/3}\right)$, $\left(-\sqrt{4/3}, \sqrt{4/3}\right)$, $\left(\sqrt{4/3}, -\sqrt{4/3}\right)$, $\left(-\sqrt{4/3}, -\sqrt{4/3}\right)$

8. **(12 points)** Find the mass of the lamina that occupies the region $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ with the density function $\rho(x, y) = e^{-x^2-y^2}$.

(1)



IN POLAR: $\begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \end{cases}$

(2)

$$\iint_D e^{-x^2-y^2}\, dA = \int_0^{2\pi}\int_0^1 e^{-r^2} r\, dr\, d\theta$$

$$= \int_0^{2\pi} 1\, d\theta \cdot \int_0^1 r e^{-r^2}\, dr$$

(3) $\displaystyle\int_0^{2\pi} 1\, d\theta = \theta\Big|_0^{2\pi} = 2\pi$

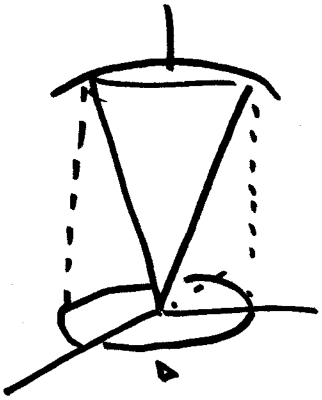
$\displaystyle\int_0^1 r e^{-r^2}\, dr = -\frac{1}{2} e^{-r^2}\Big|_0^1 = -\frac{1}{2} e^{-1^2} + \frac{1}{2} e^{-0^2} = \frac{1}{2} - \frac{1}{2} e^{-1}$

U-SUB



FINAL: $2\pi\left(\frac{1}{2} - \frac{1}{2} e^{-1}\right) = \pi\left(1 - e^{-1}\right)$

9. **(10 points)** Set up a triple integral in cylindrical coordinates to represent the volume of a solid bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $z = \sqrt{8-x^2-y^2}$. Do not evaluate.




CHANGE TO POLAR: $z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$

$z = \sqrt{8-x^2-y^2} = \sqrt{8-r^2}$

(1) TO FIND $r$: SOLVE $r = \sqrt{8-r^2}$

$r^2 = 8 - r^2$

$\Rightarrow 2r^2 = 8$

$\Rightarrow r^2 = 4$

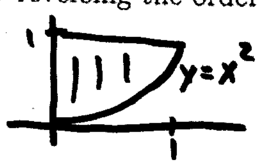
$\Rightarrow r = 2$

(2) $D = \begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \\ r \le z \le \sqrt{8-r^2} \end{cases}$

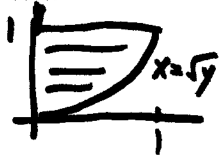
(3) $\displaystyle\int_0^{2\pi}\int_0^2\int_r^{\sqrt{8-r^2}} 1\cdot r\,dz\,dr\,d\theta$

10. **(12 points)** Sketch the region of integration for the integral $\displaystyle\int_0^1\int_{x^2}^1 \frac{\cos y}{\sqrt{y}}\,dydx$ and evaluate the integral by reversing the order of integration.

(1) $0 \le x \le 1$, $x^2 \le y \le 1$




$\longrightarrow$




$0 \le y \le 1$, $0 \le x \le \sqrt{y}$

(2) $\displaystyle\int_0^1\int_0^{\sqrt{y}} \frac{\cos y}{\sqrt{y}}\,dx\,dy$

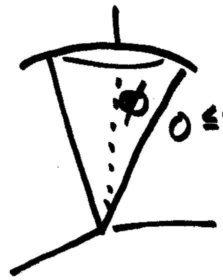
- $\displaystyle\int_0^{\sqrt{y}} \frac{\cos y}{\sqrt{y}}\,dx = \frac{\cos y}{\sqrt{y}}\,x\Big|_0^{\sqrt{y}} = \cos y$

- $\displaystyle\int_0^1 \cos y\,dy = \sin y\Big|_0^1 = \sin(1)$

11. **(10 points)** Convert the integral $\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\, dzdxdy$ into an integral in spherical coordinates. Do not evaluate.

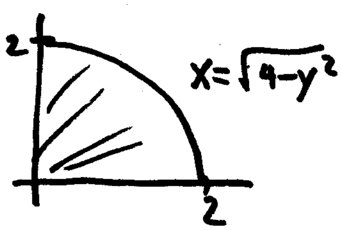
(1) $\begin{cases} 0 \le y \le 2 \\ 0 \le x \le \sqrt{4-y^2} \\ \sqrt{x^2+y^2} \le z \le \sqrt{8-x^2-y^2} \end{cases}$

(3) SKETCH



$0 \le \phi \le \pi/4$

(2)




$0 \le \theta \le \pi/2$

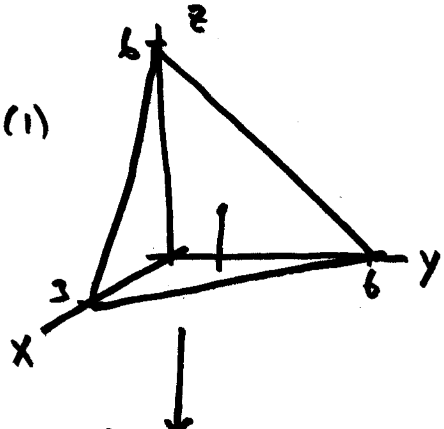
(4) $z=\sqrt{8-x^2-y^2} \Rightarrow x^2+y^2+z^2 = 8$
$\rho^2$

so $0 \le \rho \le \sqrt{8}$

(5) $\displaystyle\int_0^{\sqrt{8}} \int_0^{\pi/2} \int_0^{\pi/4} (\rho^2)^{3/2} \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$

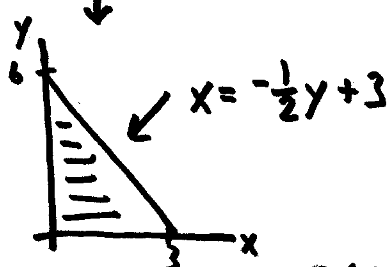
12. **(10 points)** Set up an iterated triple integral in the order of $dzdxdy$ for the volume of the solid bounded by the coordinate planes and by $2x+y+z=6$. Do not evaluate.

(1)




(3) $\displaystyle\int_0^6 \int_0^{-\frac{1}{2}y+3} \int_0^{6-2x-y} 1 \cdot dz\,dx\,dy$

(2)




$0 \le y \le 6$
$0 \le x \le -\frac{1}{2}y+3$
$0 \le z \le 6-2x-y$

13. **(15 points)** (a) Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation $u = x+y$, $v = y-2x$.

*NEED $x = g(u,v)$ AND $y = h(u,v)$

(1) $u = x+y$, $v = y-2x$ $\Rightarrow$ $2u = 2x+2y$, $v = -2x+y$

$2u+v = 3y$

$\boxed{y = \dfrac{2u+v}{3}}$

(2) $u = x+y$, $v = y-2x$ $\Rightarrow$ $u = x+y$, $-v = 2x-y$

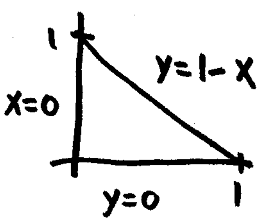
$u-v = 3x$

$\boxed{x = \dfrac{u-v}{3}}$

(3) $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} - \frac{-2}{9} = \frac{3}{9}$ OR $\frac{1}{3}$

(b) Use your answer from (a) to re-express the integral $\int_0^1 \int_0^{1-x} (x+y)^2\,(y-2x)^3\,dy\,dx$ as an iterated integral in terms of the variables $u$ and $v$. Do not evaluate.

(1) BOUNDS IN X AND Y



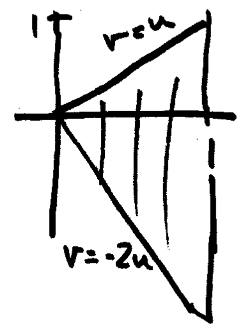

(2) CHANGE TO U AND V

$x = 0 \Rightarrow \dfrac{u-v}{3} = 0 \Rightarrow \boxed{u = v}$

$y = 0$: $\dfrac{2u+v}{3} = 0$, $\boxed{v = -2u}$

$y = 1-x \Rightarrow \dfrac{2u+v}{3} = 1 - \dfrac{u-v}{3} \Rightarrow 2u+v = 3-(u-v) \Rightarrow 2u+v = 3-u+v \Rightarrow \boxed{u = 1}$

(3)




$\begin{cases} 0 \le u \le 1 \\ -2u \le v \le u \end{cases}$

(4) $\displaystyle\int_0^1 \int_{-2u}^{u} u^2 v^3\,dv\,du$

14. **(18 points)** Consider the vector field $\mathbf{F}(x, y) = (2x \sin y)\,\mathbf{i} + (x^2 \cos y + 2y)\mathbf{j}$ in $\mathbb{R}^2$.

(a) Prove that $\mathbf{F}$ is conservative. LET $P(x,y) = 2x\sin y$ , $Q(x,y) = x^2\cos y + 2y$

$P_y = 2x\cos y$ , $Q_x = 2x\cos y$

SINCE $P_y = Q_x$ , F IS CONSERVATIVE

(b) Find a function $f$ such that $\nabla f = \mathbf{F}$. $f_x = 2x\sin y$, $f_y = x^2\cos y + 2y$

- $f = \int 2x\sin y\,dx = x^2\sin y + g(y)$
- $f_y = x^2\cos(y) + g'(y)$ SHOULD MATCH $f_y = x^2\cos y + 2y$

  SO $g'(y) = 2y \rightarrow g(y) = y^2 + k$

- $f(x,y) = x^2\sin y + y^2 + k$

(c) Use (b) to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is the curve given by $\mathbf{r}(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}, 0 \leq t \leq 1$.

(1) USE $\int_C F \cdot dr = f(r(1)) - f(r(0))$

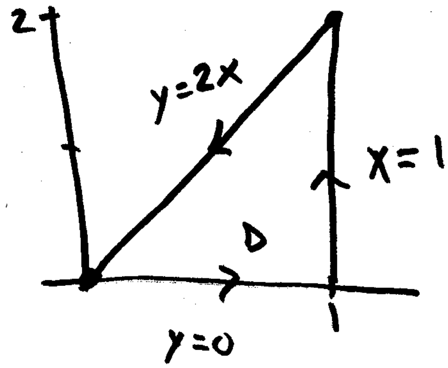
(2) $r(1) = 2i + j = \langle 2,1 \rangle$

$r(0) = i + 0j = \langle 1, 0 \rangle$

(3) $f(r(1)) - f(r(0)) = f(2,1) - f(1,0) = (4\sin 1 + 1^2) - (1\sin 0 + 0)$

$= 4\sin 1 + 1$

15. **(15 points)** Use Green's Theorem to evaluate the line integral $\oint_C \underset{P(x,y)}{\underbrace{xy^2}}dx + \underset{Q(x,y)}{\underbrace{2x^2y}}dy$, where $C$ is the triangle from $(0,0)$ to $(1,0)$ to $(1,2)$ to $(0,0)$.




(1) $\int_C xy^2dx + 2x^2y\,dy = \iint_D (Q_x - P_y)\,dA$

(2) $D = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2x \end{cases}$ $\qquad P_y = 2xy, \quad Q_x = 4xy$

(3) $\int_0^1\int_0^{2x} 4xy - 2xy \; dy\,dx$

- $\int_0^{2x} 2xy\,dy = xy^2\Big|_{y=0}^{y=2x} = 4x^3$
- $\int_0^1 4x^3dx = x^4\Big|_0^1 = \boxed{1}$