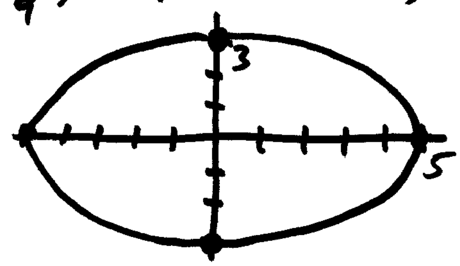
MATH 232 FINAL EXAM **Show all work for full credit** (Spring 2008)

Name (print): **Signature:** SOLUTION

1. (10pts) Find a Cartesian equation for $x = 5\sin t$, $y = 3\cos t$, and sketch the curve.

$$\frac{1}{25}x^2 + \frac{1}{9}y^2 = \frac{1}{25}(5\sin t)^2 + \frac{1}{9}(3\cos t)^2 = \underbrace{\sin^2 t + \cos^2 t}_{\text{TRIG IDENTITY}} = 1$$

SO $\frac{1}{25}x^2 + \frac{1}{9}y^2 = 1$ (ELLIPSE)



2. (10pts) Find the tangent line to the curve $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ at $(0, 1, 0)$.

✓ $t = 0$

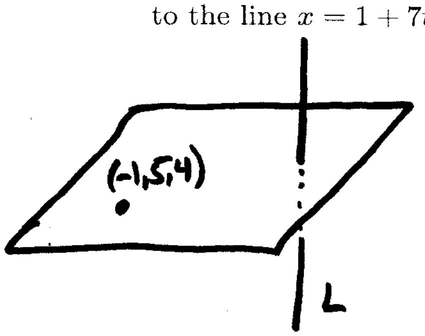
$r'(t) = \langle \cos t, -\sin t, 1 \rangle$ AND $r'(0) = \langle 1, 0, 1 \rangle$

DIRECTION VECTOR

LINE: $x = 1t + 0$, $y = 0t + 1$, $z = 1t + 0$

$\Rightarrow$ $x = t$ $y = 1$, $z = t$

3. (10pts) Find an equation of the plane through the point $(-1, 5, 4)$ and perpendicular to the line $x = 1 + 7t$, $y = t$, $z = 2 - 3t$.



$x = 1 + 7t$
$y = t$
$z = 2 - 3t$

- FOR A PLANE WE NEED POINT AND NORMAL VECTOR
- NORMAL VECTOR = DIRECTION VECTOR OF L

  $\vec{n} = \langle 7, 1, -3 \rangle$

- EQN: $7(x+1) + 1(y-5) - 3(z-4) = 0$

  $7x + 7 + y - 5 - 3z + 12 = 0$

  $7x + y - 3z = -14$

4. (10pts) Find $\lim_{(x,y)\to(0,0)} \frac{x-y^2}{x+y^2}$ or show that the limit does not exist.

ALONG $y=0$: $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x-0^2}{x+y^2} = \lim_{x\to 0} \frac{x}{x} = 1$

ALONG $x=0$: $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0-y^2}{0+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$

SINCE $1 \neq -1$, THE LIMIT DOES NOT EXIST

5. (10pts) Find the gradient and the maximum directional derivative for $f(x,y,z) = x^2yz^3$ at the point $(1,2,1)$.

- GRADIENT: $\nabla f = \langle 2xyz^3, x^2z^3, 3x^2yz^2 \rangle$
- MAX DIRECTIONAL DERIVATIVE OCCURS IN THE DIRECTION OF $\nabla f(1,2,1)$ AND HAS VALUE $|\nabla f(1,2,1)|$
- $\nabla f(1,2,1) = \langle 4, 1, 6 \rangle$
- $|\nabla f(1,2,1)| = \sqrt{4^2+1^2+6^2} = \sqrt{53}$

6. (10pts) Find parametric equations of the normal line to the surface $3x^2 + y^2 - z^2 = 4$ at the point $(1,1,0)$. LET $f(x,y,z) = 3x^2+y^2-z^2-4$ so

THE NORMAL LINE IS THE NORMAL VECTOR TO THE TANGENT PLANE.

LET'S FIND THE TANGENT PLANE

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$ WHERE $\langle a,b,c\rangle$ IS NORMAL VECTOR

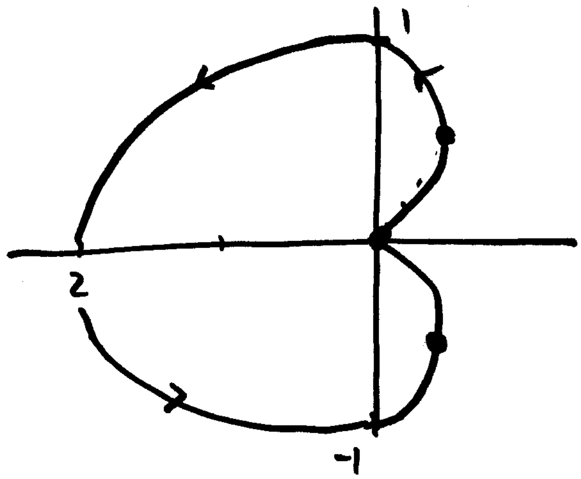
$\nabla f = \langle 6x, 2y, -2z \rangle$, $\nabla f(1,1,0) = \langle 6, 2, 0 \rangle = \langle a,b,c \rangle$

LINE: DIRECTION VECTOR: $\langle 6, 2, 0 \rangle$ POINT: $\langle 1, 1, 0 \rangle$

$x = 6t+1$, $y = 2t+1$, $z = 0$

7. (a) (5pts) Sketch the polar curve $r = 1 - \cos\theta$. (b) (10pts) Find the area of the region enclosed by the curve. FIND POINTS (r,θ) ON $r = 1 - \cos\theta$

θ	r
0	$1-\cos 0 = 1-1 = 0$
$\pi/4$	$1-\cos \pi/4 = 1-\frac{\sqrt{2}}{2}$
$\pi/2$	$1-\cos \pi/2 = 1-0 = 1$
π	$1-\cos(\pi) = 1-(-1) = 2$
$\frac{3\pi}{2}$	$1-\cos\left(\frac{3\pi}{2}\right) = 1$
$\frac{7\pi}{4}$	$1-\cos\frac{7\pi}{4} = 1-\frac{\sqrt{2}}{2}$
2π	$1-\cos 2\pi = 0$



8. (15pts) Find all critical points of $f(x, y) = x^3 + y^3 - 3xy + 9$. Then use the second derivative test to determine whether each critical point is a local maximum, local minimum, or saddle point. $f_x = 3x^2 - 3y$, $f_y = 3y^2 - 3x$, $f_{xx} = 6x$, $f_{yy} = 6y$

$f_{xy} = -3$

(1) SOLVE $f_x = 0$ AND $f_y = 0$

$\downarrow$ $\downarrow$

$3x^2 - 3y = 0$ $\quad$ $3y^2 - 3x = 0$

$\downarrow$

$y = x^2$ $\longrightarrow$ PLUG IN $y = x^2$ INTO $f_y = 0$

$3(x^2)^2 - 3x = 0$

$3x^4 - 3x = 0$

$3x(x^3 - 1) = 0$

$x = 0,\ x = 1$

(2) CRIT. POINTS $(0,0)$ AND $(1,1)$

(3) $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9$

$(0,0)$: $D = 36(0)(6) - 9 = -9 < 0$

SADDLE POINT

$(1,1)$: $D = 36(1)(1) - 9 = 27 > 1$

AND $f_{xx}(1,1) = 6(1) > 0$ SO $(1,1)$

IS A LOCAL MINIMUM

9. (15pts) Use Lagrange multipliers to find the maximum and minimum values of $f(x,y,z) = 6x + 6y - 4z$ subject to $x^2 + 3y^2 + z^2 = 1$. LET $g(x,y,z) = x^2 + 3y^2 + z^2 = 1$

- SOLVE $\begin{cases} \nabla f = \lambda g \\ g = 1 \end{cases} \Rightarrow \begin{cases} 6 = \lambda 2x \\ 6 = \lambda 6y \\ -4 = \lambda 2z \\ x^2 + 3y^2 + z^2 = 1 \end{cases}$ SOLVE FOR x, y, AND z

$$x = \frac{3}{\lambda},\ y = \frac{1}{\lambda},\ z = \frac{-2}{\lambda}$$

- PLUG $x = \frac{3}{\lambda}$, $y = \frac{1}{\lambda}$, $z = \frac{-2}{\lambda}$ INTO $x^2 + 3y^2 + z^2 = 1$

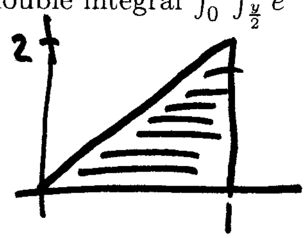
$$\frac{9}{\lambda^2} + \frac{3}{\lambda^2} + \frac{4}{\lambda^2} = 1 \Rightarrow \frac{16}{\lambda^2} = 1 \Rightarrow \lambda = -4, 4$$

- $\lambda = 4$: $x = \frac{3}{4}$, $y = \frac{1}{4}$, $z = \frac{-2}{4}$ $\qquad f(3/4, 1/4, -1/2) = 8$ MAX

$\lambda = -4$: $x = \frac{-3}{4}$, $y = \frac{-1}{4}$, $z = \frac{2}{4}$ $\qquad f(-3/4, -1/4, 1/2) = -8$ MIN

10. (15pts) Evaluate the double integral $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dx\,dy$.

SWITCH ORDER:



$y = 0$ TO $y = 2$ $\quad 0 \le y \le 2$

$x = \frac{y}{2}$ TO $x = 1$ $\quad \frac{y}{2} \le x \le 1$

CHANGE TO $\quad 0 \le x \le 1$

$\qquad\qquad\quad 0 \le y \le 2x$

$$\int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx$$

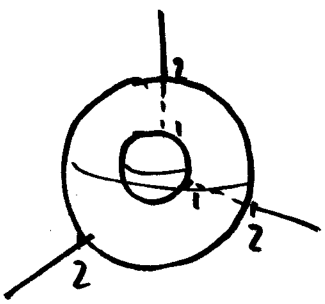
- $\displaystyle\int_0^{2x} e^{x^2}\,dy = y e^{x^2}\Big|_{y=0}^{y=2x} = 2x e^{x^2}$

- $\displaystyle\int_0^1 2x e^{x^2}\,dx = e^{x^2}\Big|_{x=0}^{x=1} = e^{1^2} - e^{0^2} = e - 1$

11. (15pts) Find the surface area for the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the circular cylinder $x^2 + y^2 = 1$.

SKIP

12. (15pts) Evaluate $\int\int\int_E \sqrt{x^2 + y^2 + z^2}\,dV$, where E is the solid region given by $1 \leq x^2 + y^2 + z^2 \leq 4$.



$$E = \begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases}$$

$$x^2 + y^2 + z^2 = \rho^2$$

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi} \sqrt{\rho^2}\, \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

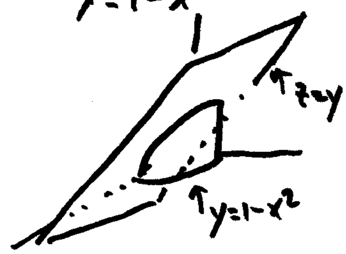
$$= \int_1^2 \rho^3 d\rho \cdot \int_0^{2\pi} 1\, d\theta \int_0^{\pi} \sin\phi\, d\phi$$

$$= \frac{1}{4}\rho^4 \Big|_1^2 \cdot \theta \Big|_0^{2\pi} \cdot -\cos\phi \Big|_0^{\pi} = \left[4 - \frac{1}{4}\right] \cdot \left[2\pi - 0\right] \cdot \left[1 + 1\right]$$

$$= 15\pi$$

13. (a) (5pts) Set up an iterated integral in the order of $dzdydx$ for the mass of the solid bounded by $z = y$, $z = 0$, and $\underbrace{x^2 = 1 - y}_{y=1-x^2}$ with $x \geq 0$, assuming the density function is $\rho(x, y, z) = x$.

$$E = \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1-x^2 \\ 0 \leq z \leq y \end{cases}$$

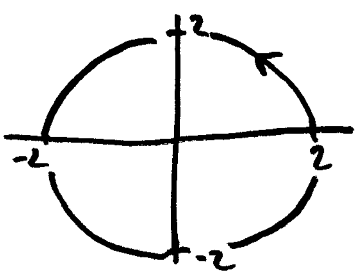


$$\int_0^1 \int_0^{1-x^2} \int_0^y \overset{\rho(x,y,z)}{x} \, dz\,dy\,dx$$

(b) (10pts) Evaluate the triple integral in (a).

- $\int_0^y x\,dz = xz\Big|_0^y = xy - 0 = xy$
- $\int_0^{1-x^2} xy\,dy = \frac{1}{2}xy^2\Big|_{y=0}^{y=1-x^2} = \frac{1}{2}x(1-x^2)^2$
- $\int_0^1 \frac{1}{2}x(1-x^2)^2 dx$ LET $u = 1-x^2$, $du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$
- $= \int_1^0 -\frac{1}{4}u^2 du = \frac{-1}{12}u^3\Big|_1^0 = \frac{-1}{12}0^3 + \frac{1}{12}1^3 = \frac{1}{12}$

14. (10pts) Evaluate the line integral $\int_C y dx - x dy$, where C is the positively oriented circle $x^2 + y^2 = 4$.



$x = 2\cos(t)$ $\quad x'(t) = -2\sin(t)$

$y = 2\sin(t)$ $\quad y'(t) = 2\cos(t)$

$$\int_C y\,dx - x\,dy = \int_0^{2\pi} 2\sin(t)\cdot -2\sin(t) - 2\cos(t)\cdot 2\cos(t)\,dt$$

$$= \int_0^{2\pi} -4\sin^2(t) - 4\cos^2(t)\,dt$$

$$= \int_0^{2\pi} -4\left(\sin^2(t) + \cos^2(t)\right) dt$$

$$= \int_0^{2\pi} -4\,dt = -4t\Big|_0^{2\pi} = -8\pi$$

15. (10pts) Evaluate the line integral $\int_C (xz + 2y)ds$, where C is the line segment from $(0, 1, 0)$ to $(1, 0, 2)$. LINE: $x = t,\ y = 1-t,\ z = 2t \quad 0 \le t \le 1$

- $xz + 2y = t(2t) + 2(1-t) = 2t^2 - 2t + 2$
- $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(1)^2 + (-1)^2 + (2)^2} = \sqrt{6}$
- $\int_0^1 f(x(t), y(t), z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_0^1 (2t^2 - 2t + 2)\sqrt{6}\, dt$

$= 2\sqrt{6}\int_0^1 t^2 - t + 1\, dt = 2\sqrt{6}\left[\frac{1}{3}t^3 - \frac{1}{2}t^2 + t\Big|_0^1\right] = 2\sqrt{6}\left(\frac{5}{6}\right) = \frac{5\sqrt{6}}{3}$

16. (a) (5pts) State the fundamental theorem for line integrals. (b) (10pts) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = y^2\mathbf{i} + (2xy + 1)\mathbf{j} + z^2\mathbf{k}$ is a conservative vector field and C is the curve $x = \cos t,\ y = \sin t,\ z = t,\ 0 \le t \le \pi$.

(a) ASSUME C IS A SMOOTH CURVE GIVEN BY VECTOR FUNCTION $r(t)$ $a \le t \le b$. LET f BE A DIFF. FUNCTION SUCH THAT ∇f IS CONT. THEN $\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$

(b) IF $F = \nabla f = \underbrace{y^2}_{f_x} i + \underbrace{(2xy+1)}_{f_y} j + \underbrace{z^2}_{f_z} k$, THEN $\quad r(\pi) = \langle -1, 0, \pi \rangle$, $r(0) = \langle 1, 0, 0 \rangle$

(1) $f = \int y^2 dx = xy^2 + g(y, z)$, (2) $f_y = 2xy + g'(y,z) = 2xy + 1$
SO $g'(y,z) = 1 \Rightarrow g(y,z) = y + h(z)$

(3) $f_z = h'(z) = z^2$ SO $h(z) = \frac{1}{3}z^3$. (4) $f = xy^2 + y + \frac{1}{3}z^3$

(5) $\int_C \nabla f \cdot dr = f(-1, 0, \pi) - f(1, 0, 0) = \left[\frac{1}{3}\pi^3\right] - [0] = \frac{1}{3}\pi^3$