EXAM III MATH 232, CALCULUS III, SPRING 2016 NAME:

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1. Evaluate

   (a) $\int_1^3 \int_0^1 (1 + 4xy)\, dx\, dy$

   (b) $\int_1^2 \int_0^3 \frac{xe^x}{y}\, dx\, dy$

   (c) $\int\int y^2\, dA$ where $D = \{(x, y) \mid -1 \leq y \leq 1, -y - 2 \leq x \leq y\}$

2. Evaluate (by changing the order of integration)

   (a) $\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$

   • (b) $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

3. Evaluate (by converting to polar)

   (a) $\int\int_D (x^2 + y^2)^{3/2}\, dA$, where D is the region in the first quadrant bounded by lines $y = 0$ and $y = \sqrt{3}x$, and the circle $x^2 + y^2 = 9$.

   • (b) $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} xy^2\, dy\, dx$

• 4. Set up but do not evaluate the triple integral to find the volume of the solid tetrahedron which is bounded by $3x + y + z = 1$ and the coordinate planes (i.e., the first octant). Evaluate.

5. Evaluate $\int\int\int_E yz\cos(x^5)\, dV$ where $E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, x \leq z \leq 2x\}$

• 6. Set up the integral $\int\int\int_E x - y\, dV$, where E is the solid that lies between cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy plane and below $z = y + 4$. Evaluate for practice.

7. Find the volume of the solid using spherical coordinates that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 4$. Hint will be given in class.

8. Evaluate the integral $\int\int_R \frac{x - 2y}{3x - y}$ by making the change of variables $u = x - 2y$ and $v = 3x - y$ where R is the parallelogram enclosed by the lines $x - 2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$.

9. Evaluate the integral $\int\int_R (x - 2y)^9(3x - y)^7\, dA$ by making the change of variables $u = x-2y$ and $v = 3x-y$ where R is the parallelogram enclosed by the lines $x-2y = 0$, $x - 2y = 4$, $3x - y = 1$, and $3x - y = 8$.

1 a) $\int_1^3\int_0^1 1+4xy \, dx\, dy$

INSIDE $\int_0^1 1+4xy\, dx = x+2x^2y\Big|_0^1 = [1+2y]-[0+0] = 1+2y$

OUTSIDE $\int_1^3 1+2y\, dy = y+y^2\Big|_1^3 = [3+9]-[1+1] = \boxed{10}$

1 b) $\int_1^2\int_0^3 \frac{xe^x}{y}\, dx\, dy$

$= \int_1^2 \frac{1}{y} dy \cdot \int_0^3 xe^x dx$

LEFT $\int_1^2 \frac{1}{y} dy = \ln|y|\Big|_1^2 = \ln 2 - \ln 1 = \ln 2$

RIGHT $\int_0^3 xe^x dx$ LET $u=x$, $dv=e^x dx$

$du=dx$, $v=e^x$

$\Rightarrow xe^x - \int e^x dx = xe^x - e^x \Big|_{x=0}^{x=3} = [3e^3-e^3]-[0-e^0]$

$= 2e^3+1$

FINAL: $(\ln 2)\cdot(2e^3+1)$

1c) $\int_{-1}^{1}\int_{-y-2}^{y} y^2 \, dx \, dy$

INSIDE: $\int_{-y-2}^{y} y^2 dx = y^2x \Big|_{x=-y-2}^{x=y}$

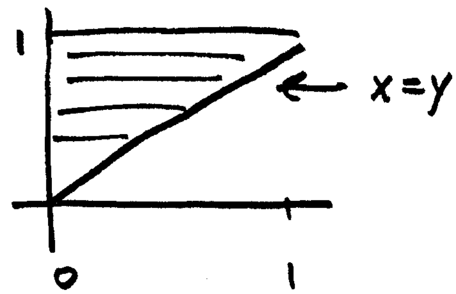
$= \left[y^2(y)\right] - \left[y^2(-y-2)\right] = y^3 + y^3 + 2y^2 = 2y^3 + 2y^2$

OUTSIDE: $\int_{-1}^{1} 2y^3 + 2y^2 dy = \frac{1}{2}y^4 + \frac{2}{3}y^3 \Big|_{y=-1}^{y=1} = \left[\frac{1}{2} + \frac{2}{3}\right] - \left[\frac{1}{2} - \frac{2}{3}\right]$

$\boxed{= \frac{4}{3}}$

2a) $\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$

SKETCH :



REORDER

$$\int_0^1 \int_0^y e^{x/y}\, dx\, dy$$

INSIDE:

$$\int_0^y e^{x/y}\, dx = y e^{x/y} \Big|_{x=0}^{x=y} = y e^{y/y} - y e^0 = ye - y$$

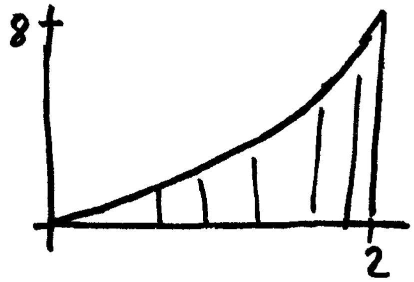
OUTSIDE

$$\int_0^1 ye - y\, dy = \tfrac{1}{2} y^2 e - \tfrac{1}{2} y^2 \Big|_{y=0}^{y=1} = \left[\tfrac{1}{2} e - \tfrac{1}{2}\right] - [0 - 0]$$

$$= \tfrac{1}{2}(e - 1)$$

2b) $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$

SKETCH



$\begin{cases} 0 \le x \le 2 \\ 0 \le y \le x^3 \end{cases}$

REORDER

$\int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$

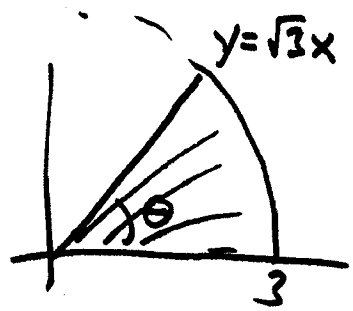
INSIDE: $\int_0^{x^3} e^{x^4}\,dy = e^{x^4} y \Big|_{y=0}^{y=x^3} = x^3 e^{x^4}$

OUTSIDE: $\int_0^2 x^3 e^{x^4}\,dx$ LET $u = x^4$, $du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$\int_0^{16} \frac{1}{4} e^u\,du = \frac{1}{4}e^{16} - \frac{1}{4}e^0 = \frac{1}{4}\left(e^{16}-1\right)$

3a) $\iint_D (x^2+y^2)^{3/2}\, dA$



$$D = \begin{cases} 0 \le r \le 3 \\ 0 \le \theta \le \pi/3 \end{cases}$$

$$\int_0^3 \int_0^{\pi/3} (r^2)^{3/2}\, r\, d\theta\, dr = \int_0^3 \int_0^{\pi/3} r^4\, d\theta\, dr$$

$$= \int_0^3 r^4\, dr \cdot \int_0^{\pi/3} 1\, d\theta$$

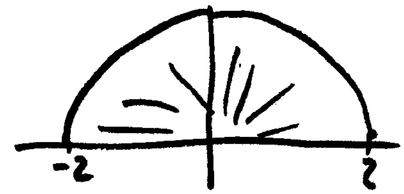
$$= \frac{1}{5} r^5 \Big|_0^3 \cdot \theta \Big|_0^{\pi/3}$$

$$= \left[\frac{243}{5} - 0\right] \cdot \left[\frac{\pi}{3} - 0\right]$$

$$= \frac{243\pi}{15}$$

3b) $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} xy^2 \, dy\,dx$

SKETCH $D = \begin{cases} -2 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \end{cases}$



$\begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le \pi \end{cases}$

$$\int_0^2\int_0^\pi (r\cos\theta)(r\sin\theta)^2 r \, d\theta \, dr = \int_0^2\int_0^\pi r^4 \cos\theta \sin^2\theta \, d\theta \, dr$$

$$= \int_0^2 r^4 dr \cdot \int_0^\pi \cos\theta \sin^2\theta \, d\theta$$

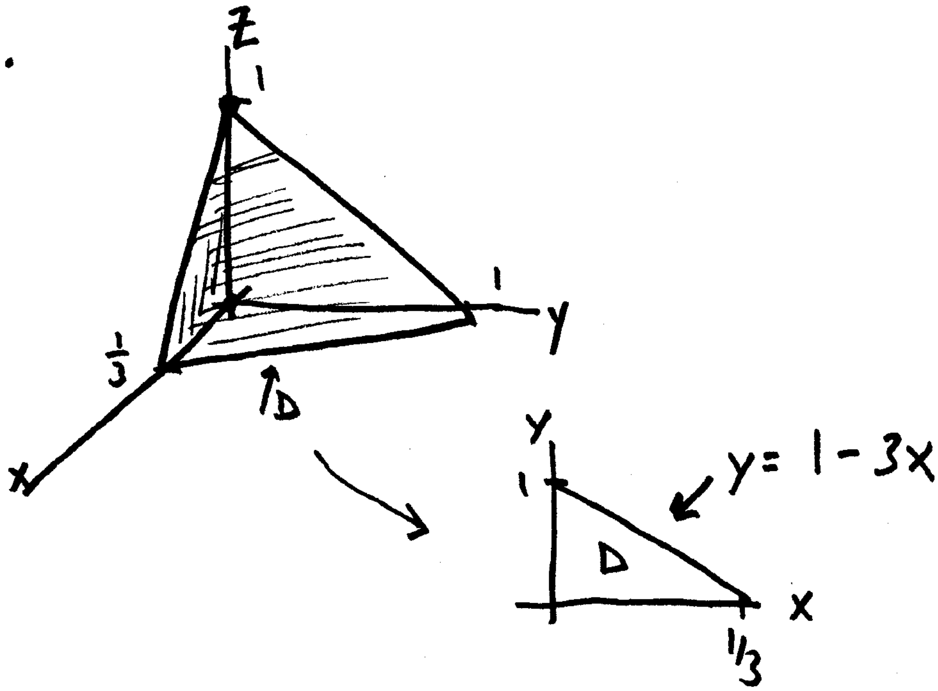
LEFT: $\int_0^2 r^4 dr = \frac{1}{5} r^5 \Big|_0^2 = \frac{32}{5}$

RIGHT: $\int_0^\pi \cos\theta \sin^2\theta \, d\theta$ LET $u = \sin\theta$, $du = \cos\theta \, d\theta$

$$\int_0^0 u^2 du = 0$$

FINAL: $\frac{32}{5} \cdot 0 = \boxed{0}$

4.



$$\int_0^{1/3}\int_0^{1-3x}\int_0^{1-3x-y} 1 \, dz\, dy\, dx$$

NOTE: FOR VOLUME OF SOME SOLID E

$$V = \iint_D \left[\int_{z=\text{BOTTOM}}^{z=\text{TOP}} 1 \, dz\right] dA$$

EVALUATE:

INSIDE: $\displaystyle\int_0^{1-3x-y} 1 \, dz = z\Big|_0^{1-3x-y} = 1-3x-y$

MIDDLE: $\displaystyle\int_0^{1-3x} 1-3x-y \, dy = y - 3xy - \tfrac{1}{2}y^2\Big|_0^{1-3x}$

$$= (1-3x) - 3x(1-3x) - \tfrac{1}{2}(1-3x)^2$$

$$= 1-3x-3x+9x^2-\frac{1}{2}(1-6x+9x^2)$$

$$= 1-3x-3x+9x^2-\frac{1}{2}+3x-\frac{9}{2}x^2$$

$$= \frac{1}{2}-3x+\frac{9}{2}x^2$$

OUTSIDE: $\int_0^{1/3} \frac{1}{2}-3x+\frac{9}{2}x^2\,dx$

$$= \frac{1}{2}x-\frac{3}{2}x^2+\frac{9}{6}x^3\Big|_0^{1/3}$$

$$= \left[\frac{1}{2}\left(\frac{1}{3}\right)-\frac{3}{2}\left(\frac{1}{3}\right)^2+\frac{9}{6}\left(\frac{1}{3}\right)^3\right]-\left[\,0\,\right]$$

$$=$$

5. $$\int_0^1\int_0^x\int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx$$

INSIDE: $$\int_x^{2x} yz\cos(x^5)\,dz = \frac{1}{2}yz^2\cos(x^5)\Big|_{z=x}^{z=2x}$$

$$= \left[\frac{1}{2}y(2x)^2\cos(x^5)\right] - \left[\frac{1}{2}y(x)^2\cos(x^5)\right]$$

$$= 2yx^2\cos(x^5) - \frac{1}{2}yx^2\cos(x^5)$$

$$= \frac{3}{2}yx^2\cos(x^5)$$

MIDDLE: $$\int_0^x \frac{3}{2}yx^2\cos(x^5)\,dx = \frac{3}{4}y^2x^2\cos(x^5)\Big|_{y=0}^{y=x}$$

$$= \left[\frac{3}{4}(x)^2\cdot x^2\cos(x^5)\right] - 0$$

$$= \frac{3}{4}x^4\cos(x^5)$$

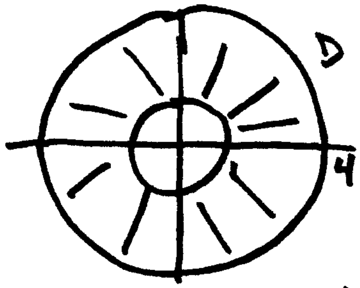
OUTSIDE: $$\int_0^1 \frac{3}{4}x^4\cos(x^5)\,dx$$ LET $u = x^5$, $du = 5x^4dx$

$\Rightarrow \frac{1}{5}du = x^4dx$

$$\Rightarrow \int_0^1 \frac{3}{20}\cos(u)\,du = \frac{3}{20}\sin(u)\Big|_0^1 = \frac{3}{20}\sin 1 - \frac{3}{20}\sin 0$$

$$= \frac{3}{20}\sin 1$$

6) SETUP IN CYLINDRICAL



$$\iint_D \left[\int_0^{y+4} x-y \, dz\right] dA$$

WRITE D IN POLAR

$$\begin{cases} 1 \le r \le 4 \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\int_0^{2\pi}\int_1^4\int_0^{r\sin\theta+4} (r\cos\theta - r\sin\theta)\, r \, dz \, dr \, d\theta$$

SOLUTION TO #6

$$\int_0^{2\pi}\int_{\phi}^{4}\int_0^{r\sin\theta+4} r^2\cos\theta - r^2\sin\theta \, dz\,dr\,d\theta$$

INSIDE: $\int_0^{r\sin\theta+4} r^2\cos\theta - r^2\sin\theta \, dz = \left(r^2\cos\theta - r^2\sin\theta\right) z \Big|_0^{r\sin\theta+4}$

$= \left(r^2\cos\theta - r^2\sin\theta\right)\left(r\sin\theta+4\right) - \left(r^2\cos\theta - r^2\sin\theta\right)(0)$

$= r^3\sin\theta\cos\theta + 4r^2\cos\theta - r^3\sin^2\theta - 4r^2\sin\theta$

MIDDLE: $\int_1^4 r^3\sin\theta\cos\theta + 4r^2\cos\theta - r^3\sin^2\theta - 4r^2\sin\theta \, dr$

(1) $\int_1^4 r^3\sin\theta\cos\theta \, dr = \frac{1}{4}r^4\sin\theta\cos\theta \Big|_1^4 = \frac{255}{4}\sin\theta\cos\theta$

(2) $\int_1^4 4r^2\cos\theta \, dr = \frac{4}{3}r^3\cos\theta \Big|_1^4 = \frac{256}{3}\cos\theta - \frac{4}{3}\cos\theta = \frac{252}{3}\cos\theta$
$= 84\cos\theta$

(3) $\int_1^4 -r^3\sin^2\theta \, dr = -\frac{1}{4}r^4\sin^2\theta \Big|_1^4 = -64\sin^2\theta + \frac{1}{4}\sin^2\theta$
$= -\frac{255}{4}\sin^2\theta$

(4) $\int_1^4 -4r^2\sin\theta \, dr = -\frac{4}{3}r^3\sin\theta \Big|_1^4 = -\frac{256}{3}\sin\theta + \frac{4}{3}\sin\theta = -84\sin\theta$

OUTSIDE: $\int_0^{2\pi} \frac{255}{4}\sin\theta\cos\theta + 84\cos\theta - \frac{255}{4}\sin^2\theta - 84\sin\theta \, d\theta$

(1) $\int_0^{2\pi} \frac{255}{4} \sin\theta \cos\theta \, d\theta = \frac{255}{8} \sin^2\theta \Big|_0^{2\pi} = 0 - 0 = 0$

(2) $\int_0^{2\pi} 84 \cos\theta \, d\theta = 84 \sin\theta \Big|_0^{2\pi} = 0 - 0 = 0$

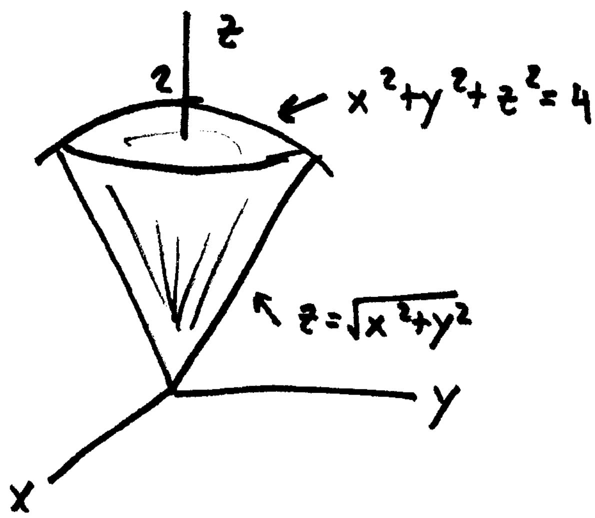
(3) $\int_0^{2\pi} -\frac{255}{4} \sin^2\theta \, d\theta = \int_0^{2\pi} -\frac{255}{8} + \frac{255}{8} \cos 2\theta \, d\theta$

$$= -\frac{255}{8}\theta + \frac{255}{16} \sin 2\theta \Big|_0^{2\pi}$$

$$= \left[-\frac{255}{8}(2\pi) + \frac{255}{16} \sin 4\pi\right] - \left[0 + 0\right]$$

$$= -\frac{255}{4}\pi$$

7)



$$\begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/4 \end{cases}$$

$$V = \int_0^2 \int_0^{2\pi} \int_0^{\pi/4} 1 \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_0^2 \rho^2 d\rho \cdot \int_0^{2\pi} 1 \, d\theta \cdot \int_0^{\pi/4} \sin\phi \, d\phi$$

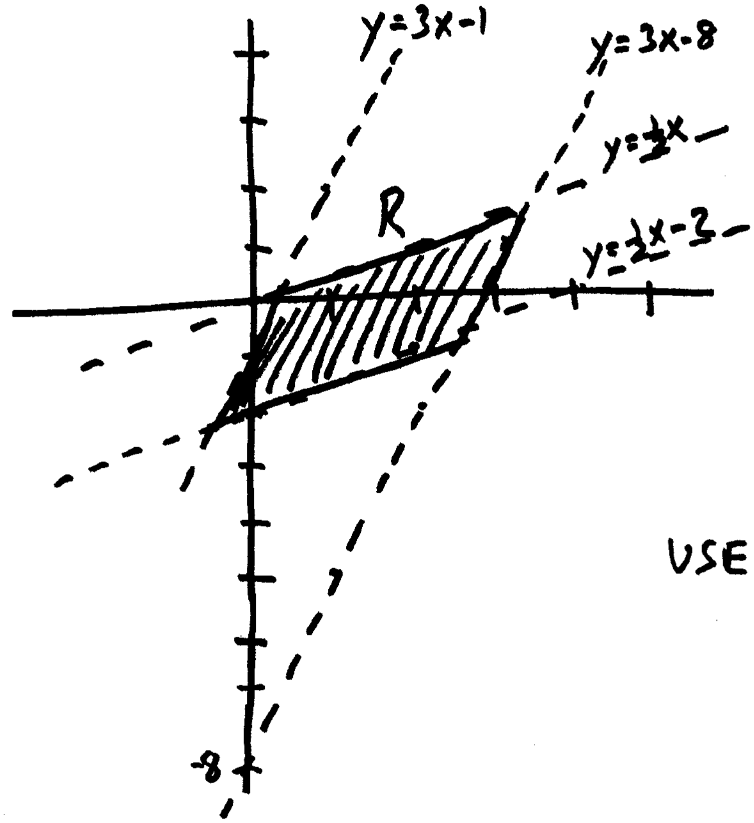
LEFT: $\int_0^2 \rho^2 d\rho = \frac{1}{3}\rho^3 \Big|_0^2 = \frac{8}{3}$

MIDDLE: $\int_0^{2\pi} 1 d\theta = \theta \Big|_0^{2\pi} = 2\pi$

RIGHT: $\int_0^{\pi/4} \sin\phi \, d\phi = -\cos(\phi) \Big|_0^{\pi/4} = -\cos(\pi/4) + \cos(0)$
$= -\frac{\sqrt{2}}{2} + 1$

FINAL: $\frac{8}{3}(2\pi)\left(1 - \frac{\sqrt{2}}{2}\right)$

8) ORIGINAL DOMAIN:



$x-2y=0 \Rightarrow y=\frac{1}{2}x$

$x-2y=4 \Rightarrow y=\frac{1}{2}x-2$

$3x-y=1 \Rightarrow y=3x-1$

$3x-y=8 \Rightarrow y=3x-8$

USE $x = \frac{2v-u}{5}$

$y = \frac{-3u+v}{5}$

TRANSFORMATION

(1) $y=\frac{1}{2}x$

$$\frac{-3u+v}{5}=\frac{1}{2}\cdot\frac{2v-u}{5}$$

$$-3u+v=\frac{1}{2}(2v-u)$$

$$-3u+v=v-\frac{1}{2}u$$

$$-\frac{5}{2}u=0$$

$$\boxed{u=0}$$

(2) $y = \frac{1}{2}x - 2$

$$\Rightarrow \quad \frac{-3u+v}{5} = \frac{1}{2} \cdot \frac{2v-u}{5} - 2$$

$$\Rightarrow \quad -3u+v = \frac{1}{2}(2v-u) - 10$$

$$\Rightarrow \quad -3u+v = v - \frac{1}{2}u - 10$$

$$-\frac{5}{2}u = -10$$

$$\boxed{u = 4}$$

(3) $y = 3x - 1$

$$\Rightarrow \quad \frac{-3u+v}{5} = 3 \cdot \frac{2v-u}{5} - 1$$

$$\Rightarrow \quad -3u+v = 3(2v-u) - 5$$

$$\Rightarrow \quad -3u+v = 6v - 3u - 5$$

$$-5v = -5$$

$$\boxed{v = 1}$$

(4) $y = 3x - 8$

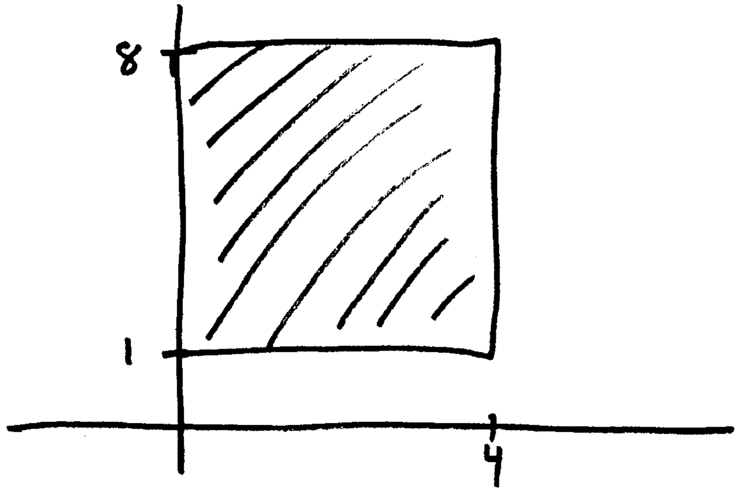
$$\Rightarrow \quad \frac{-3u+v}{5} = 3 \cdot \frac{2v-u}{5} - 8$$

$$\Rightarrow \quad -3u+v = 3(2v-u) - 40$$

$$-3u+v = 6v - 3u - 40$$

$$-5v = -40 \qquad \boxed{v = 8}$$

NEW DOMAIN:



$$D = \begin{cases} 0 \le u \le 4 \\ 1 \le v \le 8 \end{cases}$$

JACOBIAN FOR $x = \frac{2v - u}{5}$ AND $y = -\frac{3u + v}{5}$

$$|J| = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} = \left| \frac{-1}{25} - \frac{6}{25} \right| = \left| \frac{-7}{25} \right| = \frac{7}{25}$$

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_0^4 \int_1^8 \frac{u}{v}\left|\frac{-7}{25}\right| dv\,du$$

$$= \frac{7}{25}\int_0^4 u\,du \cdot \int_1^8 \frac{1}{v}\,dv$$

$$= \frac{7}{25} \cdot \left[\frac{1}{2}u^2\Big|_0^4\right] \cdot \left[\ln|v|\Big|_1^8\right]$$

$$= \frac{7}{25} \cdot \frac{16}{2} \cdot \left(\ln 8 - \ln 1\right)$$

$$= \frac{56}{25}\ln 8$$

#9) USES THE SAME TRANSFORMATION AS #8

$$\iint_R (x-2y)^9(3x-y)^7\,dA = \int_0^4\int_1^8 u^9 v^7 \left|\frac{-7}{25}\right| dv\,du$$

$$= \frac{7}{25}\int_0^4 u^9\,du \cdot \int_1^8 v^7\,dv$$

$$= \frac{7}{25}\cdot\left[\frac{1}{10}u^{10}\Big|_0^4\right]\cdot\left[\frac{1}{8}v^8\Big|_1^8\right]$$

$$= \frac{7}{25}\cdot\underbrace{\frac{4^{10}}{10}\cdot\left[\frac{8^8}{8}-\frac{1}{8}\right]}_{\text{SORRY FOR WEIRD \#}}$$