(1) FIND ARC LENGTH OF $r(t) = \langle \sin 2t, -2t, -\cos 2t \rangle$ FOR $1 \leq t \leq 4$

$$x = \sin 2t, \quad \frac{dx}{dt} = 2\cos 2t,$$

$$y = -2t, \quad \frac{dy}{dt} = -2$$

$$z = -\cos 2t, \quad \frac{dz}{dt} = 2\sin 2t$$

$$L = \int_1^4 \sqrt{(2\cos 2t)^2 + (-2)^2 + (2\sin 2t)^2}\, dt$$

$$= \int_1^4 \sqrt{4\cos^2(2t) + 4 + 4\sin^2(2t)}\, dt$$

$$= \int_1^4 \sqrt{4\left(\cos^2 2t + \sin^2 2t\right) + 4}\, dt$$

$$= \int_1^4 \sqrt{8}\, dt$$

$$= \sqrt{8}\, t \Big|_1^4$$

$$= \sqrt{8}(4) - \sqrt{8}(1)$$

$$= 3\sqrt{8}$$

#2. $a(t) = \langle \cos t, -3, 3t^2 \rangle$. FIND $v(t)$ IF $v(0) = \langle -2, 5, 3 \rangle$

- $v(t) = \int (\cos t)i - 3j + 3t^2 k \, dt$, WHERE $v(0) = -2i + 5j +$

$$= (\sin t)i - 3tj + t^3 k + C$$

- $v(0) = -2i + 5j + 3k = (\sin 0)i - 3(0)j + 0^3 k + C$

$$-2i + 5j + 3k = 0 - 0 + 0 + C$$

$$\Rightarrow \quad C = -2i + 5j + 3k$$

- $v(t) = (\sin t)i - 3tj + t^3 k + \overbrace{(-2i + 5j + 3k)}^{C}$

$$= (\sin t - 2)i + (-3t + 5)j + (t^3 + 3)k$$

3. SHOW $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+2y^2}$ DOES NOT EXIST

TRY DIFFERENT PATHS

$$x=0: \quad \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{2(0)y}{0^2+2y^2} = \lim_{y\to 0} \frac{0}{2y^2} = 0$$

$$y=0: \quad \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{2x(0)}{x^2+2(0)^2} = \lim_{y\to 0} \frac{0}{x^2} = 0$$

$$y=x: \quad \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{2x(x)}{x^2+2(x)^2} = \lim_{x\to 0} \frac{2x^2}{3x^2} = \frac{2}{3}$$

SINCE DIFFERENT PATHS LEAD TO DIFFERENT LIMIT VALUES, THE LIMIT DOES NOT EXIST.

#4. EVALUATE $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{10x^2+10y^2}}$ USING POLAR COORDINATES

- $x = r\cos\theta,\quad y = r\sin\theta,\quad x^2+y^2=r^2$
- $$\lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{10(r\cos\theta)^2+10(r\sin\theta)^2}}$$

$$= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{\sqrt{10r^2(\cos^2\theta+\sin^2\theta)}}$$

$$= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{\sqrt{10}\cdot r}$$

$$= \lim_{r\to 0} \frac{r\cos\theta\sin\theta}{\sqrt{10}}$$

$$= \frac{0}{\sqrt{10}}\cdot\text{(BOUNDED)}$$ NOT $\pm\infty$

$$= \underline{\underline{0}}$$

#5 LET $z = yz + x\ln(y)$. USE IMPLICIT DIFFERENTIATION TO FIND $\frac{dz}{dx}$

$$\frac{d}{dx}\left[z\right] = \frac{d}{dx}\left[yz + x\ln(y)\right]$$

$$\frac{dz}{dx} = y \cdot 1 \cdot \frac{dz}{dx} + 1 \cdot \ln y$$

$$\frac{dz}{dx} - y\frac{dz}{dx} = \ln y$$

$$\frac{dz}{dx}\left(1-y\right) = \ln y$$

$$\frac{dz}{dx} = \frac{\ln y}{1-y}$$

OR

LET $f(x,y) = yz + x\ln y - z = 0$

$$\frac{dz}{dx} = \frac{-F_x}{F_z} = \frac{-\ln y}{y-1} = \frac{\ln y}{1-y}$$

#6 FIND EQUATION OF TANGENT ~~EQ.~~ PLANE AT (1, -2, 1) TO THE SURFACE $f(x,y) = 3x^2 + 2x - y^2$.

(a) FORMULA: $z - z_0 = f_x(1,-2)(x - x_0) + f_y(1,-2)(\cancel{y} - y_0)$

$$z - 1 = f_x(1,-2)(x-1) + f_y(1,-2)(y+2)$$

$f_x = 6x + 2$; $f_x(1,-2) = 6(1) + 2 = 8$

$f_y = -2y$; $f_y(1,-2) = -2(-2) = 4$

$$z - 1 = 8(x-1) + 4(y+2)$$

$$z = 8x - 8 + 4y + 8 + 1$$

TANGENT PLANE: $z = 8x + 4y + 1$

(b) USE LINEARIZATION: $L(x,y) = 8x + 4y + 1$

$$f(1.1, -1.8) \approx L(1.1, -1.8) = 8(1.1) + 4(-1.8) + 1$$

$$= 2.6$$

EXACT VALUE: $f(1.1, -1.8) = 3(1.1)^2 + 2(1.1) - (-1.8)^2 = 2.59$

#7. LET $f(x,y) = \tan^{-1}(x^2+y^2)$, $x = s\ln(t)$, $y = te^s$. USE CHAIN RULE TO FIND $\frac{\partial z}{\partial t}$ AT $t=1, s=0$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = \left[\frac{1}{1+(x^2+y^2)^2}\cdot 2x\right]\cdot\left[\frac{s}{t}\right] + \left[\frac{1}{1+(x^2+y^2)^2}\cdot 2y\right]\cdot\left[e^s\right]$$

AT $t=1, s=0 \Rightarrow x = 0\ln(1) = 0$

$y = 1\cdot e^0 = 1$

$$\left.\frac{\partial z}{\partial t}\right|_{\substack{t=1\\ s=0}} = \left[\frac{1}{1+(0^2+1^2)^2}\cdot 2(0)\right]\cdot\left[\frac{0}{1}\right] + \left[\frac{1}{1+(0^2+1)^2}\cdot 2(1)\right]e^0$$

$$= [0]\cdot[0] + [1]\cdot[1]$$

$$= 1$$

#8 FIND DIRECTIONAL DERIVATIVE OF $z = x^2 e^{-y}$ AT $(3,0)$ IN THE DIRECTION OF $3i + 4j = \vec{v}$

- NEED v AS A UNIT VECTOR: $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{3i + 4j}{5} = \frac{3}{5}i + \frac{4}{5}j$

- FIND GRADIENT: ∇f

$$\nabla f = \langle 2xe^{-y}, -x^2e^{-y} \rangle$$

$$\nabla f(3,0) = \langle 2 \cdot 3 e^0, -3^2 e^0 \rangle$$

$$= \langle 6, -9 \rangle$$

- DIRECTIONAL DERIVATIVE IN DIRECTION OF $\vec{u}$

$$D_{\vec{u}} f(3,0) = \nabla f(3,0) \cdot \vec{u}$$

$$= \langle 6, -9 \rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5} \rangle$$

$$= 6 \cdot \frac{3}{5} + -9 \cdot \frac{4}{5}$$

$$= -\frac{18}{5} \text{ OR } -3.6$$

#9 FIND LOCAL EXREMA AND SADDLE POINT(S) OF

$f(x,y) = x^2 - xy + y^2 + 9x - 6y + 10$

- SOLVE $f_x = 0$ AND $f_y = 0$ FOR CRITICAL POINTS

$f_x = 2x - y + 9 = 0;$ $\quad f_y = -x + 2y - 6 = 0$

SYSTEM OF EQUATIONS

$$\begin{cases} 2x - y = -9 \\ -x + 2y = 6 \end{cases}$$

$$\Rightarrow \begin{cases} 2x - y = -9 \\ -2x + 4y = 12 \end{cases}$$

$$3y = 3$$

$$y = 1$$

PLUG INTO $2x - y = -9$ TO FIND x

$$2x - 1 = -9$$

$$2x = -8$$

$$x = -4$$

CRITICAL POINT AT $(-4, 1)$

(2) FIND $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

#9 CONT

$f_{xx} = 2$

$f_{xy} = -1$

$f_{yx} = -1$

$f_{yy} = 2$

$$D = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2 \cdot 2 - (-1)^2$$

$$= 4 - 1$$

$$= 3 \quad > 0$$

SINCE $D > 0$ AND $f_{xx} > 0$, $(-4, 1)$ IS A LOCAL MIN

NOTE: OUR TEST QUESTION IS HARDER. OURS WILL BE MORE LIKE EX. 2 FROM OUR 14.7 NOTES

#10 USE LAGRANGE MULTIPLIERS TO FIND THE MAX AND MIN VALUES OF $f(x,y) = x^2y$ SUBJECT TO $x^2+y^2=1$.

(1) FIND ∇f: $\quad \nabla f = \langle 2xy,\ x^2 \rangle$

(2) FIND ∇g: $\quad \nabla g = \langle 2x,\ 2y \rangle$

(3) SOLVE $\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases}$: $\quad \begin{cases} 2xy = \lambda \cdot 2x \\ x^2 = \lambda \cdot 2y \\ x^2+y^2 = 1 \end{cases}$

$2xy = \lambda 2x$ GIVES $\quad 2xy - 2\lambda x = 0$

$$2x(y-\lambda) = 0$$

SO EITHER $x=0$ OR $\lambda = y$

(4) SUPPOSE $x=0$ PLUG $x=0$ INTO $x^2+y^2=1$

$$0^2+y^2=1$$
$$y^2=1$$
$$y=\pm 1$$

$\boxed{\begin{matrix}(0,1)\\(0,-1)\end{matrix}}$

(5) SUPPOSE $y=\lambda$. THEN $x^2=\lambda 2y$ GIVES US $x^2=2y^2$

PLUG $x^2=2y^2$ INTO $x^2+y^2=1$ TO GET

$$2y^2+y^2=1$$
$$3y^2=1$$
$$y=\pm\sqrt{1/3}$$

#10 CONT.

IF $y = \pm\sqrt{1/3}$ THEN

$$x^2 = 2\left(\pm\sqrt{1/3}\right)^2$$

$$x^2 = 2\left(\tfrac{1}{3}\right)$$

$$x^2 = \tfrac{2}{3}$$

$$x = \pm\sqrt{2/3}$$

(6) POSSIBLE POINTS: $(0,1)$, $(0,-1)$, $\left(-\sqrt{2/3}, \sqrt{1/3}\right)$

$\left(-\sqrt{2/3}, -\sqrt{1/3}\right)$, $\left(\sqrt{2/3}, -\sqrt{1/3}\right)$, $\left(\sqrt{2/3}, \sqrt{1/3}\right)$

(7) EVALUATE

$$f(0,1) = 0^2(1) = 0$$

$$f(0,-1) = 0^2(-1) = 0$$

$$f\left(-\sqrt{2/3}, \sqrt{1/3}\right) = \left(-\sqrt{2/3}\right)^2\left(\sqrt{1/3}\right) = \frac{2}{3\sqrt{3}}$$

$$f\left(-\sqrt{2/3}, -\sqrt{1/3}\right) = \left(-\sqrt{2/3}\right)^2\left(-\sqrt{1/3}\right) = -\frac{2}{3\sqrt{3}}$$

$$f\left(\sqrt{2/3}, \sqrt{1/3}\right) = \left(\sqrt{2/3}\right)^2\left(\sqrt{1/3}\right) = \frac{2}{3\sqrt{3}}$$

$$f\left(+\sqrt{2/3}, -\sqrt{1/3}\right) = \left(\sqrt{2/3}\right)^2\left(-\sqrt{1/3}\right) = \frac{-2}{3\sqrt{3}}$$

MAX IS $\frac{2}{3\sqrt{3}}$ AND MIN IS $\frac{-2}{3\sqrt{3}}$

#11 FIND MAX AND MIN OF $f(x,y) = xy$ SUBJECT TO $4x^2+y^2=8$

(1) FIND ∇f: $\nabla f = \langle y, x \rangle$

(2) FIND ∇g: $\nabla g = \langle 8x, 2y \rangle$

(3) SOLVE $\begin{cases} \nabla f = \lambda \nabla g \\ g = 8 \end{cases}$

$$\Rightarrow \left\{ \begin{array}{l} y = 8x\lambda \\ x = 2y\lambda \\ 4x^2+y^2=8 \end{array} \right\} \longrightarrow \quad y = 8x\lambda \Rightarrow \lambda = \frac{y}{8x}$$

PLUG $\lambda = \frac{y}{8x}$ INTO $x = 2y\lambda$ TO GET $x = 2y\left(\frac{y}{8x}\right)$

$$\Rightarrow 4x^2 = y^2$$

(4) PLUG $y^2 = 4x^2$ INTO $4x^2+y^2=8$

$$\to 4x^2+4x^2=8$$

$$\to 8x^2=8$$

$$\to x^2=1$$

$$\to x=\pm 1$$

USING $y^2=4x^2$ WE GET $y^2=4 \to y=\pm 2$

(5) POSSIBLE POINTS: $(-1,-2), (-1,2), (1,2), (1,-2)$

(6) EVALUATE:

$f(-1,-2)=2, \quad f(-1,2)=-2, \quad f(1,2)=2, \quad f(1,-2)=-2$

MAX IS 2 MIN IS -2