Derivatives and Integrals

Definition 1: Derivative Formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(f \pm g) = f' \pm g'$$

$$\frac{d}{dx}(kx) = k$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$(f\ g)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x) \cdot e^{f(x)}$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{1}{f(x)} \cdot f'(x)$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x\ \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x\ \cot x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$$

Definition 2: Integral Formulas

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

$$\int \frac{1}{kx+b}\ dx = \frac{1}{k}\ln|kx+b| + C$$

$$\int e^x\ dx = e^x + C$$

$$\int e^{kx}\ dx = \frac{1}{k}e^{kx} + C$$

$$\int a^x\ dx = \frac{a^x}{\ln a} + C$$

$$\int a^{kx}\ dx = \frac{1}{k\ln a}a^{kx} + C$$

$$\int \ln x\ dx = x\ln x - x + C$$

$$\int \cos x\ dx = \sin x + C$$

$$\int \cos(bx)\ dx = \frac{1}{b}\sin(bx) + C$$

$$\int \sin x\ dx = -\cos x + C$$

$$\int \sin(bx)\ dx = -\frac{1}{b}\cos(bx) + C$$

$$\int \sec^2 x\ dx = \tan x + C$$

$$\int \csc^2 x\ dx = -\cot x + C$$

$$\int \sec x \tan x\ dx = \sec x + C$$

$$\int \csc x \cot x\ dx = -\csc x + C$$

$$\int \tan x\ dx = \ln|\sec x| + C$$

$$\int \cot x\ dx = \ln|\sin x| + C$$

$$\int \csc x\ dx = \ln|\csc x - \cot x| + C$$

$$\int \sec x\ dx = \ln|\sec x + \tan x| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\ dx = \sec^{-1}(x) + C$$

$$\int -\frac{1}{x\sqrt{x^2-1}}\ dx = \csc^{-1}(x) + C$$

Steps 1: Sketching a Paremetric Curve

1. Make a t-table with columns for t, x, and y.
2. Choose t values (from the domain) like $t = -1, 0, 1, 2, ...$ if the equations are rational or polynomial-ish.
3. Choose t values (from the domain) like $t = 0, \pi/4, \pi/2, \pi$, etc., if the equations are trigonometric.
4. Plot and connect the points (note the direction with arrows)

Definition 3: First Derivative of a Parametric Curve

Suppose f and g are differentiable functions where $x = f(t)$ and $y = g(t)$. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ provided } \frac{dx}{dt} \neq 0$$

Definition 4: Second Derivative of a Parametric Curve

Suppose f and g are differentiable functions where $x = f(t)$ and $y = g(t)$. Then the second derivative is

$$\frac{dy}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Definition 5: Finding the Equation of the Tangent Line

Let $x = x(t)$ and $y = y(t)$. The equation of the tangent line at $t = k$ is

$$y - y_1 = m(x - x_1)$$

where $m = \left.\frac{dy}{dx}\right|_{t=k}$, $x_1 = x(k)$, and $y_1 = y(k)$. Note: (x_1, y_1) might already be given.

Definition 6: Horizontal and Veritical Tangents

$x = x(t)$ and $y = y(t)$ has a **Horizontal Tangent Line** when

$$\frac{dy}{dt} = 0 \text{ provided } \frac{dx}{dt} \neq 0$$

And a **Vertical Tangent Line** when

$$\frac{dx}{dt} = 0 \text{ provided } \frac{dy}{dt} \neq 0$$

Formula 1: Arc Length

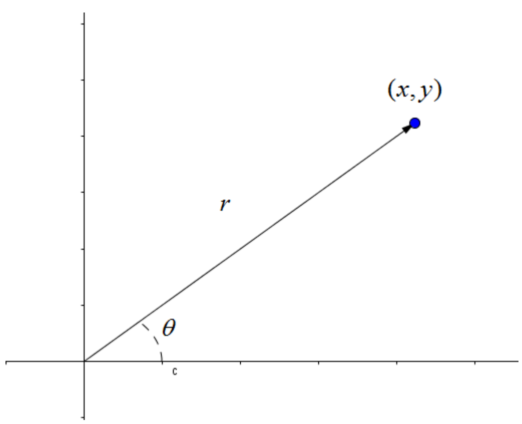
If C is described by $x = f(t)$ and $y = g(t)$ on $\alpha \leq t \leq \beta$ and are continuous and C is traversed exactly once as t increases, then

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Polar

Formula 2: Fundemental Formula for Polar Coordinates

Given the following following point (x, y)



We have the following relationships

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta)$$

$$r^2 = x^2 + y^2 \text{ and } \tan(\theta) = \frac{y}{x}$$

Steps 2: Sketching a Polar Curve

1. Make a table with columns for θ, r, and $P(r, \theta)$.
2. Choose θ values (from the domain) like $\theta = 0, \pi/3, \pi/4, \pi/2, 3\pi/4, 4\pi/3, \pi$... etc.
3. Try to plot at least 6 polar points. It's usually enough to see the pattern.
4. Plot and connect the points (note the direction with arrows)

Definition 7: Derivative of Polar Curves

Let $r = f(\theta)$.

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\dfrac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

Definition 8: Finding the Equation of the Tangent Line

Let $x = r\sin(\theta)$, $y = r\sin(\theta)$, and $r = f(\theta)$. The equation of the tangent line at $\theta = k$ is

$$y - y_1 = m(x - x_1)$$

where $m = \left.\dfrac{dy}{dx}\right|_{\theta=k}$. Once you have r and θ use $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to find x_1 and y_1.

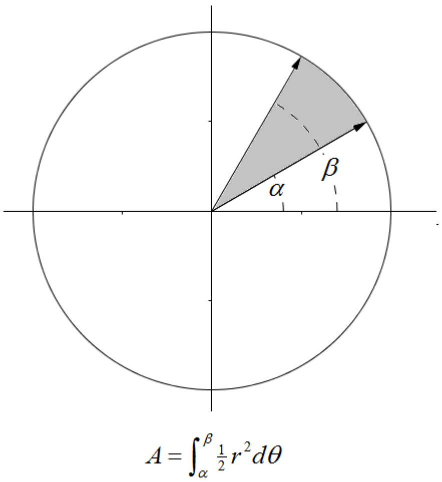
Definition 9: Horizontal and Vertical Tangents

$r = f(\theta)$ has a **Horizontal Tangent** when $\dfrac{dy}{d\theta} = 0$, provided $\dfrac{dx}{d\theta} \neq 0$.

$r = f(\theta)$ has a **Vertical Tangent** when $\dfrac{dx}{d\theta} = 0$, provided $\dfrac{dy}{d\theta} \neq 0$.

Area and Lenghts in Polar

Definition 10: Area of a Polar Region



The area for finding area enclosed under a polar curve is

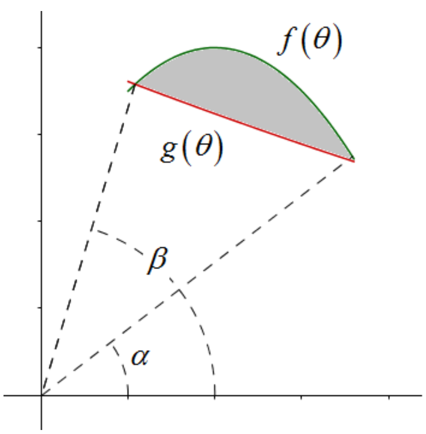
$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$

Trig Identities you will need are

$$\sin^2(\theta) = \frac{1}{2}\left(1 - \cos(2\theta)\right)$$

$$\cos^2(\theta) = \frac{1}{2}\left(1 + \cos(2\theta)\right)$$

Definition 11: Area Between Polar Curves



$$A = \int_\alpha^\beta \frac{1}{2}\left(f(\theta)\right)^2 - \frac{1}{2}\left(g(\theta)\right)^2 \, d\theta$$

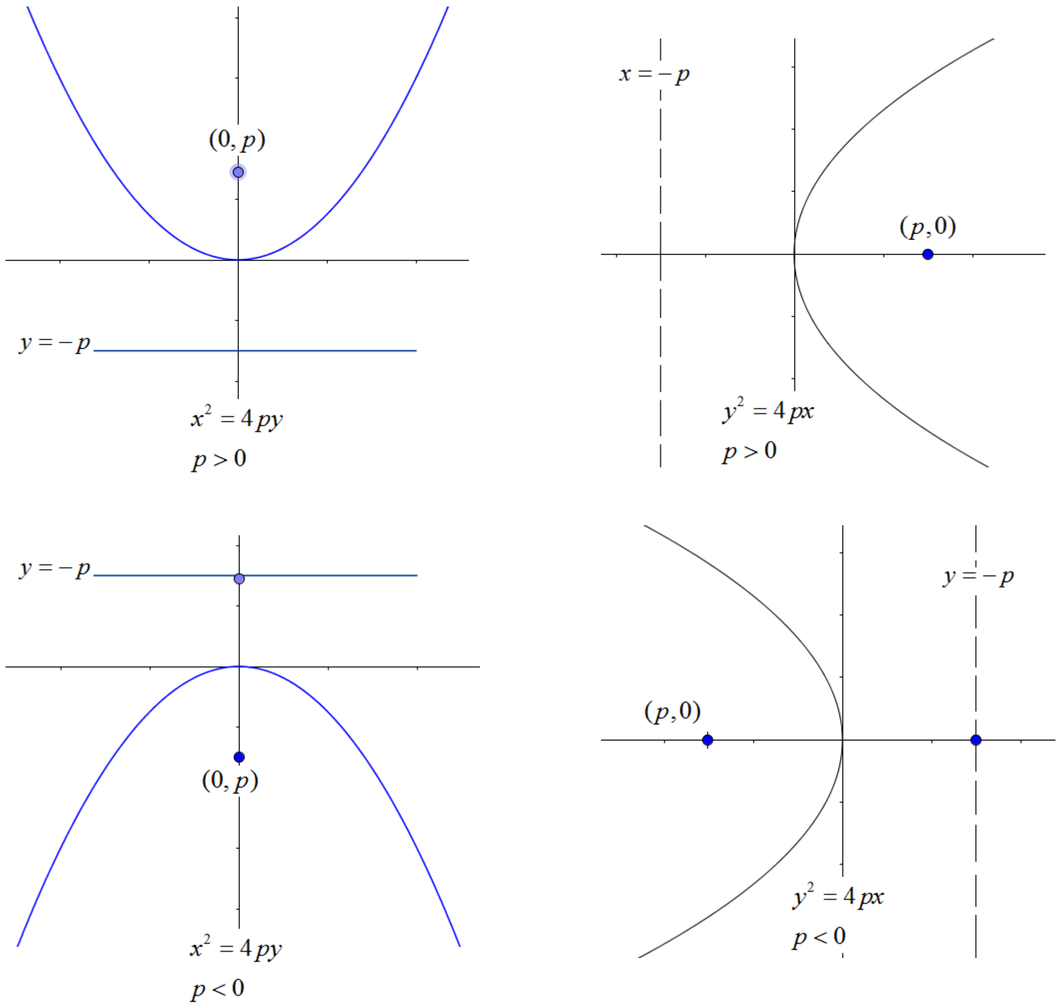
Definition 12: Arc Length with Polar Curves

Let $r = f(\theta)$. The length of r on $\alpha \le \theta \le \beta$ is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta$$

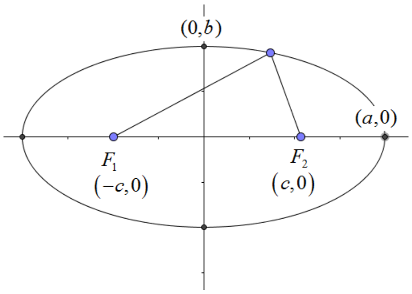
Conics

Definition 13: Parabola



Definition 14: Ellipse

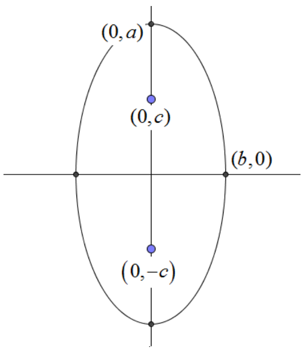
Horizontal Ellipse: $a \geq b$



Formula: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$c^2 = a^2 - b^2$

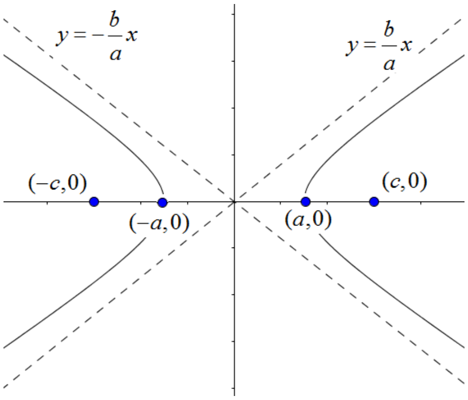
Vertical Ellipse: $a \geq b$

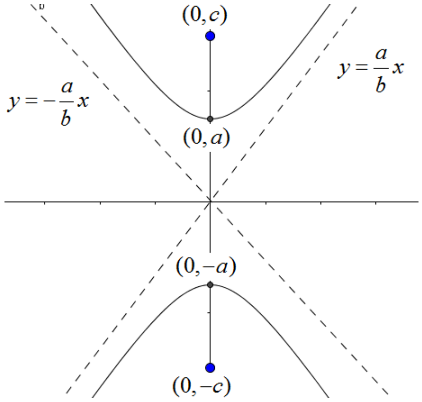


Formula: $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

$c^2 = a^2 - b^2$

Definition 15: Hyperbola





Formula: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$c^2 = a^2 + b^2$

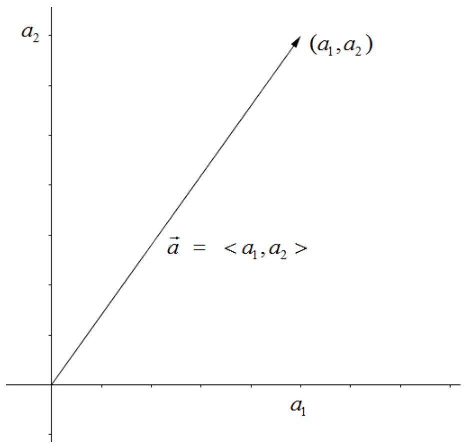
Formula: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

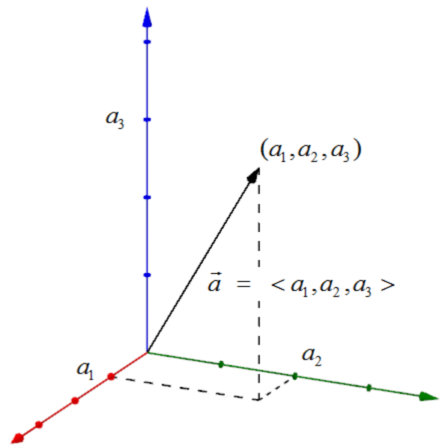
$c^2 = a^2 + b^2$

Chapter 12 - Vectors

Definition 16: Vectors Componenets

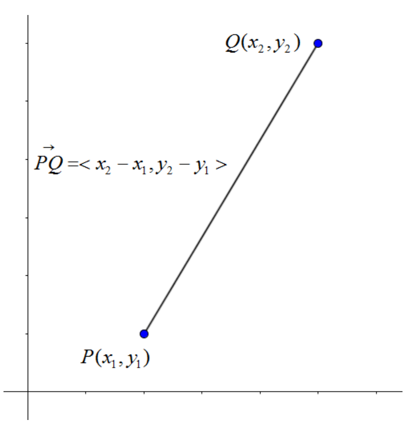
Let $\vec{a}$ be a vector defined by $\vec{a} =< a_1, a_2 >$ or $\vec{a} =< a_1, a_2, a_3 >$. a_1, a_2 and a_3 are called the components of vector $\vec{a}$.

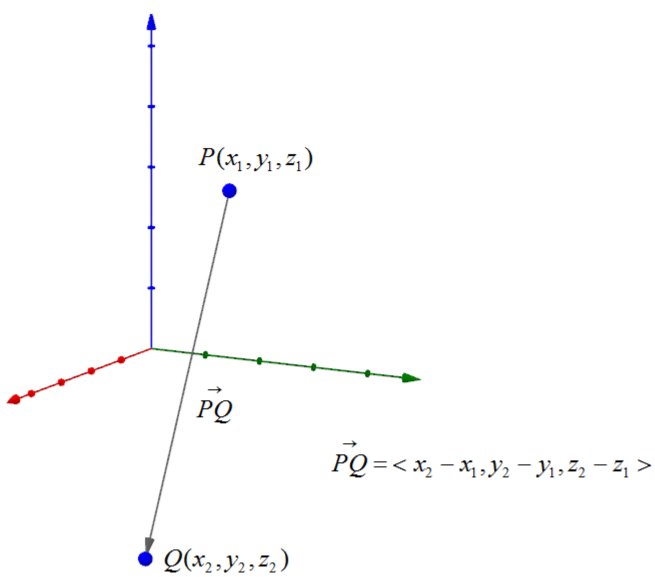




The components are the displacement from the initial point to its terminal.

Definition 17: Creating a Vector from Two Points





Definition 18: Vector Magnitude (length)

Let $\vec{a} =< a_1, a_2 >$, then the magnitude is $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$

Let $\vec{a} =< a_1, a_2, a_3 >$, then the magnitude is $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Definition 19: Vector Addition/Subtraction, Scalar Multiplication

For 2D: Let $\vec{a} =< a_1, a_2 >$, $\vec{b} =< b_1, b_2 >$, and c be a scalar, then

$$\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2 >$$

$$\vec{a} - \vec{b} =< a_1 - b_1, a_2 - b_2 >$$

$$c\vec{a} =< ca_1, ca_2 >$$

For 3D: Let $\vec{a} =< a_1, a_2, a_3 >$, $\vec{b} =< b_1, b_2, b_3 >$, and c be a scalar, then

$$\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 >$$

$$\vec{a} - \vec{b} =< a_1 - b_1, a_2 - b_2, a_3 - b_3 >$$

$$c\vec{a} =< ca_1, ca_2, ca_3 >$$

Definition 20: Unit Vector

A unit vector is a vector with length 1. If $\vec{a}$ is any vector, then

$$\frac{\vec{a}}{|\vec{a}|} \text{ is a unit vector}$$

To find a vector with the direction of $\vec{a}$ with length L

$$\vec{v} = \frac{L}{|\vec{a}|}\vec{a}$$

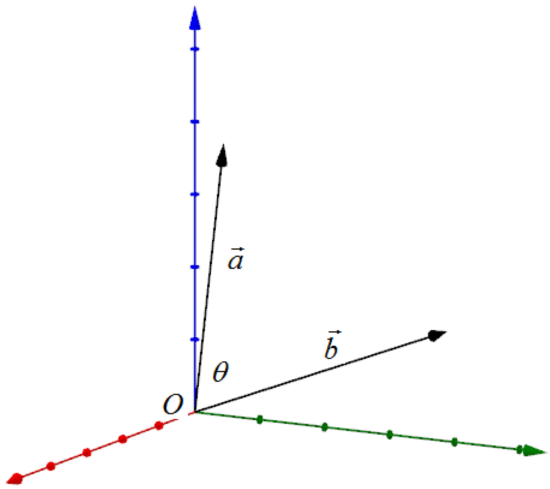
Definition 21: The Dot Product

Let $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$. Then the **Dot Product** is

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Note: The dot product is a scalar (NOT ANOTHER VECTOR)

Theorem 1



Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$. Then

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta) \text{ or } \cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Note: Use this if you want to find the angle between two vectors.

Definition 22: Orthogonal

Vectors $\vec{a}$ and $\vec{b}$ are Orthogonal or Perpendicular if $\vec{a}\cdot\vec{b} = 0$

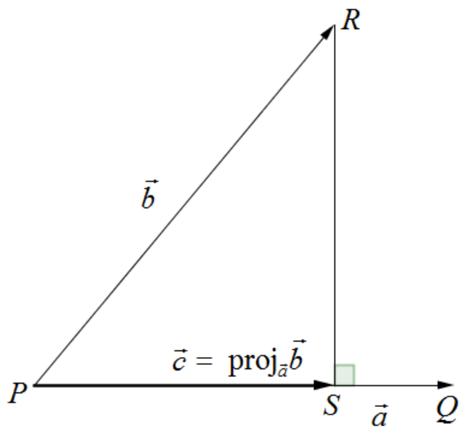
If $\vec{a}\cdot\vec{b} > 0$, the angle is acute.

If $\vec{a}\cdot\vec{b} = 0$, the angle is right.

If $\vec{a}\cdot\vec{b} < 0$, the angle is obtuse.

Definition 23: Vector Projection of $\vec{b}$ onto $\vec{a}$

It's much easier to visualize a vector projection in 2D than 3D. Let $\vec{a} = \vec{PQ}$, $\vec{b} = \vec{PR}$, and $\vec{c} = \vec{PS}$. Vector $\vec{c}$ is called the vector projection of $\vec{b}$ onto $\vec{a}$. Think of vector $\vec{c}$ as the shadow of $\vec{b}$ on $\vec{a}$ if you shined a light straight down over $\vec{b}$.



Vector Projection of $\vec{b}$ onto $\vec{a}$:

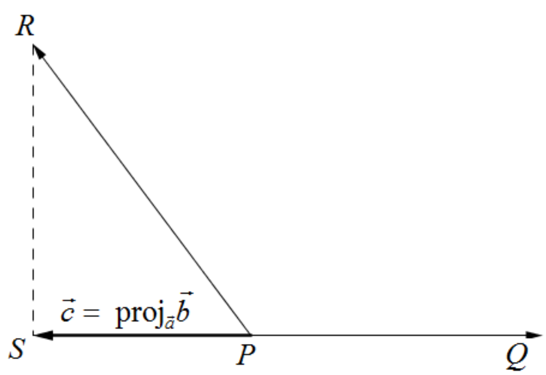
$$\vec{c} = \text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Scalar Projection of $\vec{b}$ onto $\vec{a}$

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

You can think of $\text{comp}_{\vec{a}}\vec{b}$ as the length of $\vec{c}$ with a $\pm$ to determine direction.

If the angle between vectors $\vec{a}$ and $\vec{b}$ is greater than 90 degrees, the picture would look like this:



In the above graph $\text{comp}_{\vec{a}}\vec{b}$ is negative.

Definition 24: The Cross Product (Easier)

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

Please note the $(-)$ sign on the second determinate.

Theorem 2

If θ is the angle between vectors $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\theta)$$

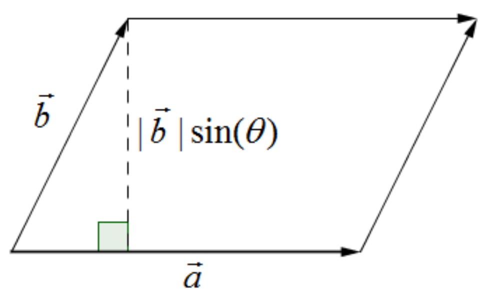
Parallel: If two vectors are parallel the angle between them is $\theta = 0$. And since $\sin(0) = 0$ it follows that

$$\vec{a} \times \vec{b} = 0 \text{ if } \vec{a} \text{ and } \vec{b} \text{ are parallel}$$

We also know that $\vec{a}$ and $\vec{b}$ are parallel if $\vec{a} = \lambda\vec{b}$ where λ is a scalar.

Definition 25: Applicaton of the Cross Product

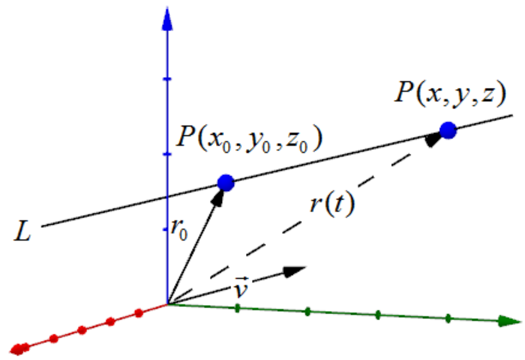
Consider the following parallelogram formed by vectors $\vec{a}$ and $\vec{b}$



The area of this parallelogram is $|\vec{a}||\vec{b}|\sin(\theta)$. By the theorem above we know this is also $|\vec{a} \times \vec{b}|$. It follows that

$$|\vec{a} \times \vec{b}| = \text{ area of parallelogram}$$

Definition 26: Vector Equation of a Line L



Let L be a line in three-dimensional space. $P(x, y, z)$ is an artibrary point on L. $P(x_0, y_0, z_0)$ is a specific point on L. r_0 is the vector that connects to $P(x_0, y_0, z_0)$. $r(t)$ is the vector that connects to a point on L. And $\vec{v}$ is the position vector that is parallel to L.

The vector equation for a line is three dimensions space is

$$\vec{r}(t) = \vec{v}t + \vec{r_0}$$

$$\vec{r}(t) =< a, b, c > t+ < x_0, y_0, z_0 >$$

where $\vec{v} =< a, b, c >=< x_1 - x_0, y_1 - y_0, z_1 - z_0 >$ and t is a parameter.

Definition 27: Parametric Equations of a Line L

Parametric equations for a line through point (x_0, y_0, z_0) and parallel to the direction vector $\vec{v} =< a, b, c >$ are

$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

Definition 28: How to Find the Direction Vector $\vec{v}$

1. You can find the direction vector $\vec{v} =< a, b, c >$ of any of the three equations for a line (vector, parametric, syymemtric).

2. Parallel to another line L_2? Use the direction vector on L_2.

3. Given two points P and Q? Then your direction vector $\vec{v} = \vec{PQ}$

4. Remember that two lines are parallel if their direction vectors $\vec{v_1}$ and $\vec{v_2}$ are proportional.

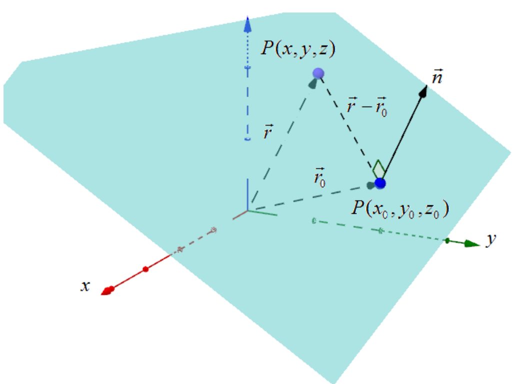
$$\vec{v_1} = \lambda \vec{v_2}$$

5. Perpendicular to another line L_2? If L_2 has direction vector $\vec{v_2}$ then your direction vector $\vec{v_1}$ must satisfy

$$\vec{v_1} \cdot \vec{v_2} = 0$$

6. Perpendicular to a plane? Use the normal vector of the plane as $\vec{v}$.

Definition 29: Planes



To create a plane you need two things: an initial point (x_0, y_0, z_0) and a vector $\vec{n}$ orthongonal to the plane.

Let $P(x_0, y_0, z_0)$ be a point on a plane with normal vector $\vec{n} =< a, b, c >$. **Vector Equation of the Plane**

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

$$\vec{n} \cdot < x, y, z > = \vec{n} \cdot < x_0, y_0, z_0 >$$

Scalar Equation of the Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

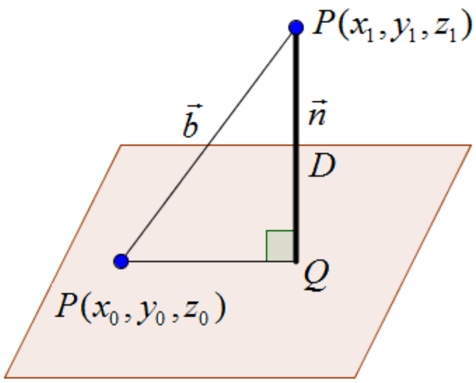
Definition 30: Find plane given three points

Suppose you have three points on a plane P, Q, and R. To find the normal vector use

$$\vec{n} = \vec{PQ} \times \vec{PR}$$

Definition 31: Shortest Distance Between and Point and a Plane

et $P(x_1, y_1, z_1)$ be a point in three dimensional space (not on the plane), $P(x_0, y_0, z_0)$ be a point on the plane, and let $\vec{n} =< a, b, c >$ be a vector normal to the plane. Then the shortest distance from $P(x_1, y_1, z_1)$ to the plane is



$$D = \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Chapter 13 - Vector Functions

Definition 32: Vector Functions

A vector function has the form

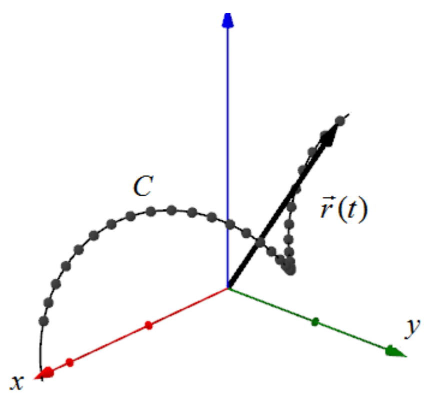
$$\vec{r}(t) =< f(t), g(t), h(t) >= f(t)i + g(t)j + h(t)k$$

where $f(t)$, $g(t)$, and $h(t)$ are called the component functions of $\vec{r}$.

The domain of $\vec{r}(t)$ are all the values of t that work for $f(t)$, $g(t)$, and $h(t)$.

Definition 33: Space Curves

Suppose f, g, and h are continuous functions on a domain D. Then the set of all points (x, y, z) in space where $x = f(t)$, $y = g(t)$, and $z = h(t)$ as t varies throughout D is called a Space Curve C.



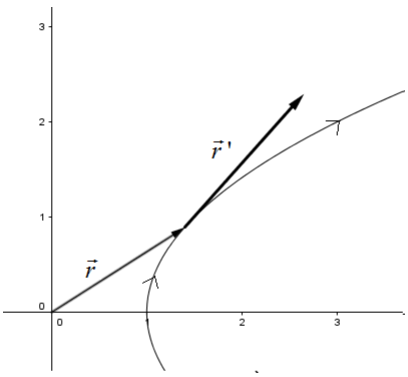
C is traced out by the tip of the vectors from $\vec{r}(t)$.

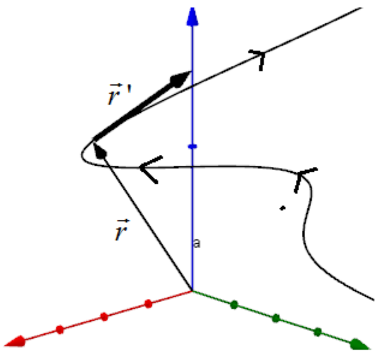
Definition 34: Derivative of a Vector Function

Given the vector function $\vec{r} =< f(t), g(t), h(t) >$, the derivative $\vec{r}'(t)$ is

$$\vec{r}'(t) =< f'(t), g'(t), h'(t) >$$

$$\vec{r}'(t) = f'(t)i + g'(t)j + h'(t)k$$





If $\vec{r'}$ is the tangent vector then $\dfrac{\vec{r'}}{|\vec{r'}|}$ is **Unit Tangent Vector**.

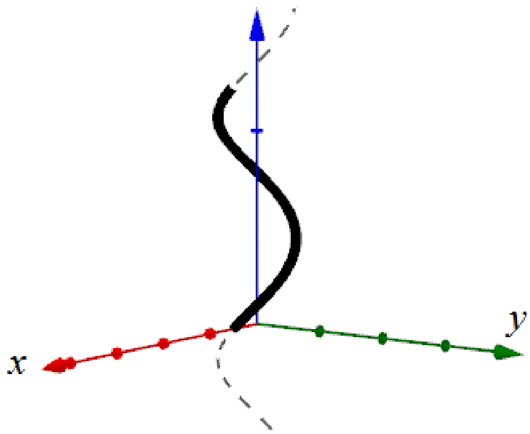
Definition 35: Integration of Vector Functions

Let $\vec{r}(t) = f(t)i + g(t)j + h(t)k =< f(t), g(t), h(t) >$. Then

$$\int_a^b \vec{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right] i + \left[\int_a^b g(t)\,dt\right] j + \left[\int_a^b h(t)\,dt\right] k$$

Definition 36: Arc (Path) Length

Let $\vec{r}(t) =< f(t), g(t), h(t) >$ or $x = f(t)$, $y = g(t)$, $z = h(t)$, $a \le t \le b$, the length of the curve is

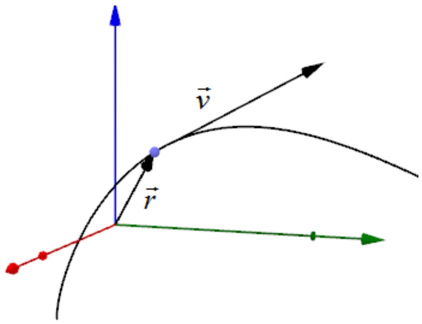


$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,dt$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Definition 37

Let $\vec{r}(t)$ be the position vector.



$\vec{v}(t) = \vec{r'}(t)$ is the velocity vector and points in the diretion of the tangent vector.

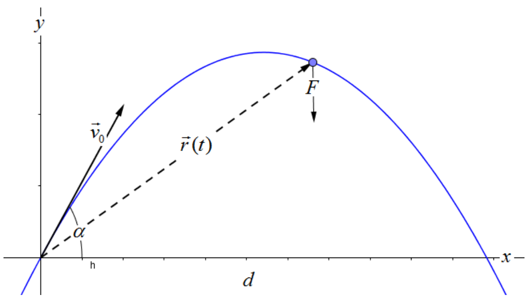
The speed of the object at time t is the magnitude of $\vec{v}$.

$$\vec{s}(t) = |\vec{v}(t)|$$

The acceleration of the object at time t is

$$\vec{a}(t) = \vec{v'}(t) = \vec{r''}(t)$$

Definition 38: Parametric Equations for Trajectory



Initial Position: $\vec{r}_0$

Initial Velocity: $\vec{v}_0$

$$\vec{r} = (\vec{v}_0 \cos(\alpha))i + \left[\vec{r}_0 + \vec{v}_0 \sin(\alpha)t - \frac{1}{2}gt^2\right] j$$

$$\text{Horizontal Distance: } x(t) = (\vec{v}_0 \cos(\alpha))t$$

$$\text{Vertical Distance: } y(t) = \vec{r}_0 + (\vec{v}_0 \sin(\alpha))t - \frac{1}{2}gt^2$$

Chapter 14 - Multivariable Functions

Theorem 3

If $f(x,y)$ is continuous at (a,b), then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Definition 39: Limits when $f(x,y)$ is not defined at (a,b)

1. Try factoring. It's how we dealt with $\dfrac{0}{0}$ in single variable calculus.

   NOTE: You cannot use L'Hospitals Rule when you have more than one variable.

2. Try multiple paths that lead to the point (a,b). Hope two different paths lead to two different values. This means the limit does not exist

3. If you see $x^2 + y^2$, you may want to change to polar.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta)$$

Theorem 4: Squeeze Theorem

Let $f(x,y) \le g(x,y) \le h(x,y)$ in a disk around (a,b) and $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L$. Then

$$\lim_{(x,y)\to(a,b)} g(x,y) = L$$

Definition 40: Notation for Partial Derivatives

Let $z = f(x,y)$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = D_y f$$

Definition 41: Higher Order Partial Derivatives

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

f_{xy} and f_{yx} are called mixed partial derivatives.

Theorem 5: Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If f_{xy} and f_{yx} are both continuous, then

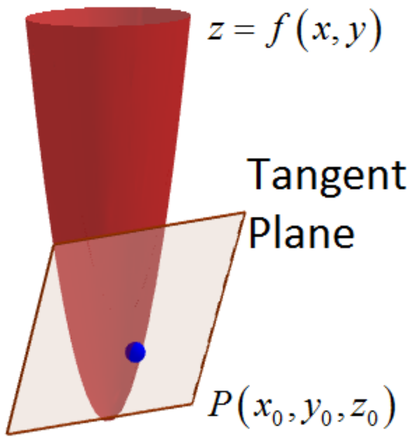
$$f_{xy} = f_{yx}$$

In fact changing the order of partial differentiation will not matter. For example,

$$f_{xxy} = f_{xyx} = f_{yxx}$$

Definition 42: Equation of a Tangent Plane

Suppose a surface S has the equation $z = f(x,y)$ such that f_x and f_y are continuous and let $P(x_0, y_0, z_0)$ be a point on S. Then the equation for the tangent plane to the surface $z = f(x,y)$ at P is



$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

where $f_x = \dfrac{\partial z}{\partial x}$ and $f_y = \dfrac{\partial z}{\partial y}$

NOTE: Sometimes the surface is given implicitly $F(x,y,z) = 0$. For example,

$$4x^3 - 2xy + yz^2 - 4 = 0$$

This means you need to use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$
$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Definition 43: Linear Approximation

The tangent plane $L(x,y) = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) + z_0$ is also called the linear approximation. We can use to to approximate z values near $P(x_0, y_0)$.

Definition 44: The Chain Rule, Case 1: One Parameter

Suppose that $z = f(x,y)$ is differentiable in x and y where $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then z is differentiable and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Definition 45: The Chain Rule, Case 2: Two Parameters

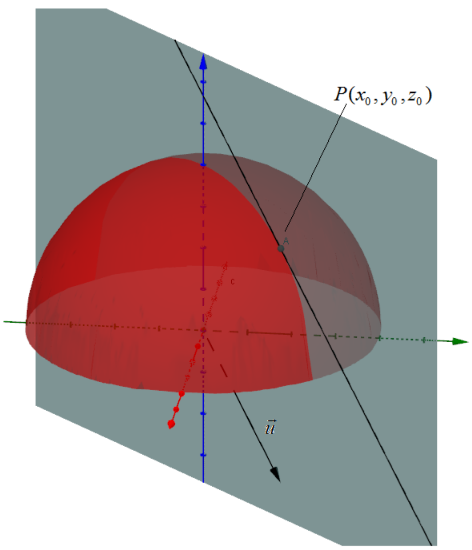
Suppose $z = f(x, y)$ is differentiable and $x = g(s, t)$ and $y = h(s, t)$ are differentiable. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Definition 46: Directional Derivative and Gradient Vector

If f is differentiable in x and y then f has a directional derivative in the direction of any unit vector $\vec{u} =< a, b >$ and



$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

or

$$D_{\vec{u}} f(x, y) = \nabla f \cdot \langle a, b \rangle$$

where

$$\nabla f = \langle f_x, f_y \rangle$$

The directional derivative will give us the slope of the tangent line T to the curve at the point $P(x_0, y_0, z_0)$ in the direction of $\vec{u}$.

Theorem 6

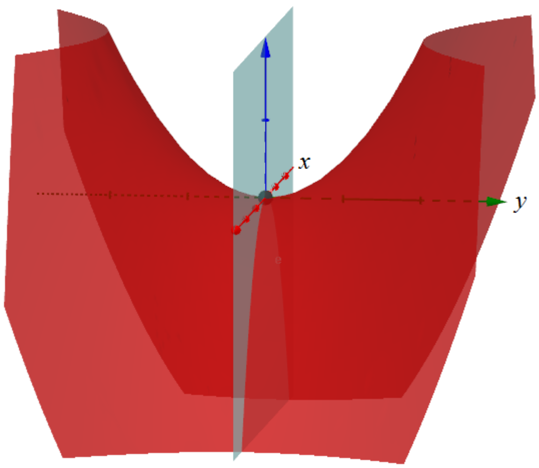
Suppose f is a differentiable function of 2 or 3 variables. The max value of the directional derivative $D_{\vec{u}} f = |\nabla f|$ and it's in the direction of the gradient vector of ∇f.

Theorem 7

If f has a local maximum or minimum at (a,b) and f_x and f_y exist, then $f_x(a,b)=0$ AND $f_y(a,b)=0$

Like in calculus I, it means all **potential max/mins** must occur when $f_x(a,b)=0$ and $f_y(a,b)=0$. Call these **Critical Points**.

Definition 47: Saddle Point at (a,b)



The point can neither be a local maximum nor a local minimum.

We call the point (a,b) a **saddle point**.

Definition 48: Second Derivative Test

Assume the second partial derivatives of f are continuous on a disk with center (a,b). Suppose $f_x(a,b)=0$, $f_y(a,b)=0$ and

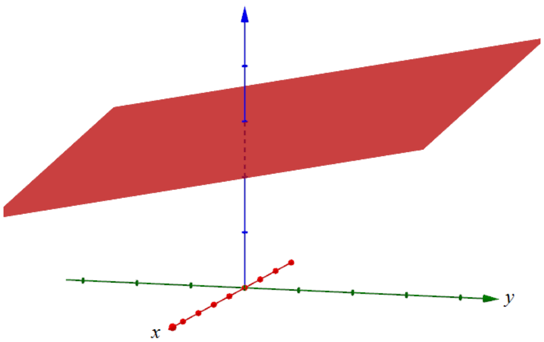
$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

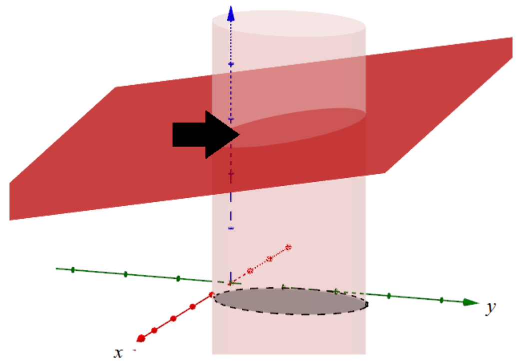
$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

1. If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum.
2. If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum.
3. If $D<0$, then $f(a,b)$ is neither a maximum nor minimum (Saddle Point)
4. If $D=0$, the test is inconclusive.

Steps 3: Method of Lagrange Multipliers

Find the absolute max and min of $f(x, y)$ subject to the contraints $g(x, y) = k$ provided $\nabla g \neq 0$.





1. Find all x and y such that

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y), & \lambda \text{ is Lagrange Multiplier} \\ g(x,y) = k \end{cases}$$

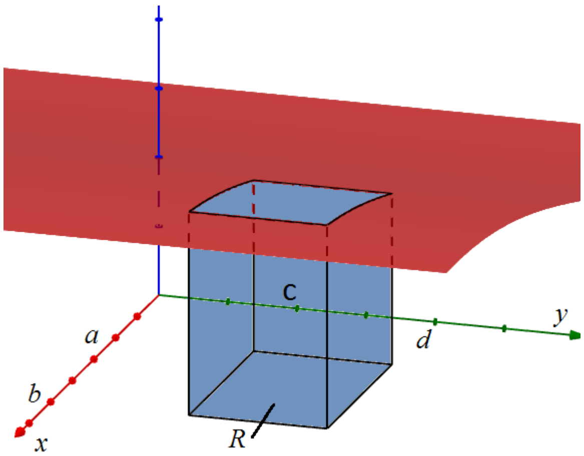
2. Then evaluate $f(x, y)$ at all points (x, y) found above.

3. The largest of these values is the absolute max.

4. The smallest of these values is the absolute min.

Chapter 15 - Multiple Integrals

Definition 49: Integral of Function of Two Variables



We want the volume under the surface S over the rectangular region

$$R = \{(x,y) | a \leq x \leq b, c \leq y \leq d\}$$

$$V = \int_a^b \int_c^d f(x,y)\ dy\ dx$$

Definition 50: Iterated Integrals - Fubini's Theorem

Let $f(x,y)$ be a function over $R = \{(x,y) | a \leq x \leq y, c \leq y \leq d\}$. Fubini's Theorem states that

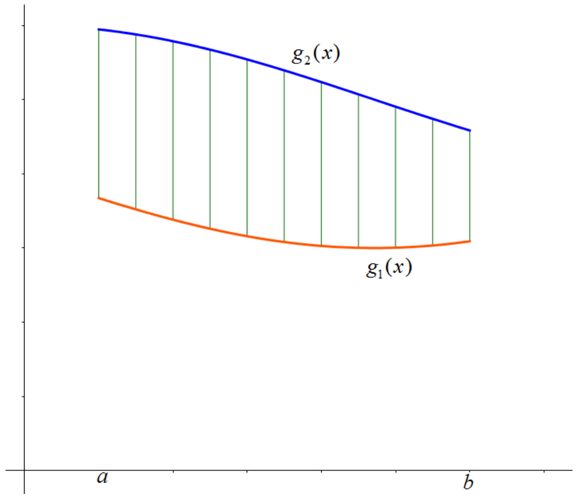
$$\int \int_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy$$

Definition 51: Special Case

If $f(x,y) = g(x)h(y)$ on $R = [a,b] \times [c,d]$ then

$$\int \int_R f(x,y)\ dA = \int \int g(x)h(y)\ dx\ dy = \int_a^b g(x)\ dx \cdot \int_c^d h(y)\ dy$$

Definition 52: Vertically Simple - Type 1

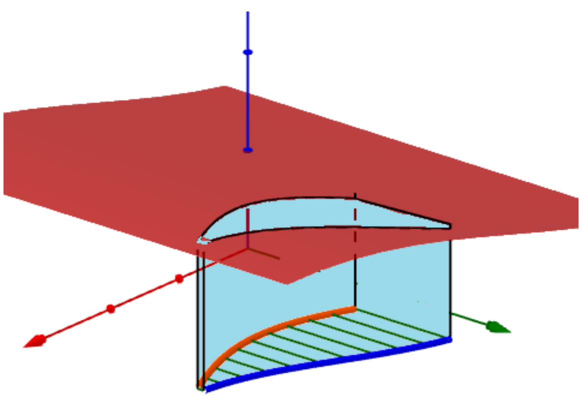


A vertically simple region (called Type 1) is a region where every vertical line drawn share the same upper function and bottom function $g_2(x)$ and $g_1(x)$

This region can be represented as such:

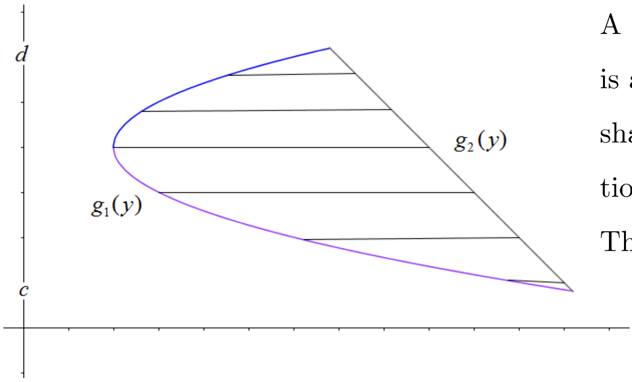
$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

This let's us evaluate the volume by the following



$$V = \int\int_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Definition 53: Horizontally Simple - Type 2

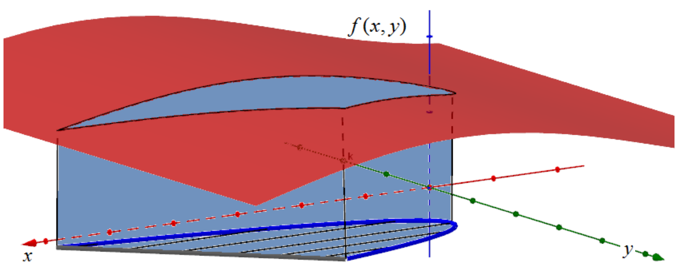


A horizontally simple region (called Type 2) is a region where every horizontal line drawn share the same right function and left function $g_2(y)$ and $g_1(y)$

This region can be represented as such:

$$D = \{(x, y) \mid c \le y \le d,\ g_1(y) \le x \le g_2(y)\}$$

This let's us evaluate the volume by the following



$$V = \int\int_D f(x, y)\, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y)\, dx\, dy$$

Definition 54: Common Questions

1. Sketch the volume given by the double integral

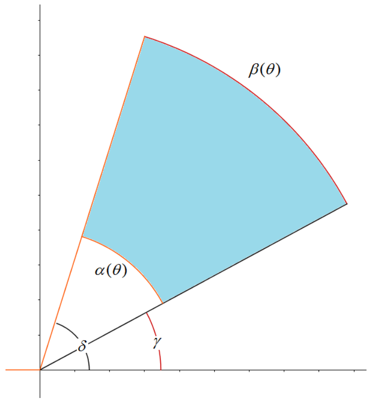
$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} f(x, y)\, dx\, dy$$

2. Evaluate an integral where you must first change the order of integration

$$\int_0^1 \int_x^1 \sin y^2\, dy\, dx \quad \text{to} \quad \int_0^1 \int_0^y \sin y^2\, dx\, dy$$

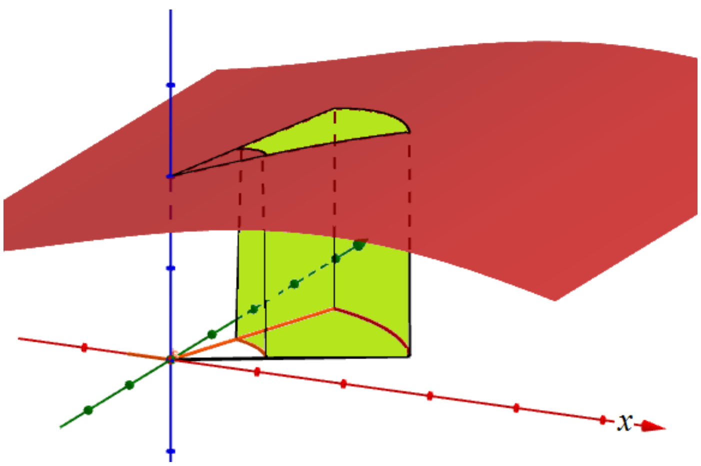
Definition 55: Radially Simple

A region that is radially simple is satisfies the following inequalities:



$$D = \{(r, \theta) \mid \gamma \leq \theta \leq \delta,\ \alpha(\theta) \leq r \leq \beta(\theta)\}$$

Definition 56: Polar Change for Double Integrals



$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$\int\int_D f(x,y)\, dA = \int_\gamma^\delta \int_{\alpha(\theta)}^{\beta(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Definition 57: Finding Area of Region Using Double Integrals

If $f(x,y) = 1$, then

$$\int\int_D 1\, dx\, dy = \text{ Area of } D$$

If you rewrite D so it's in polar, then

$$\text{Area of } D = \int_\alpha^\beta \int_0^{h(\theta)} 1r\, dr\, d\theta$$

Suppose a lamina occures a region D of the xy plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D.

Definition 58: Mass

$$\text{mass } = m = \int\int_D \rho(x, y)\, dA$$

Definition 59: Center of Mass

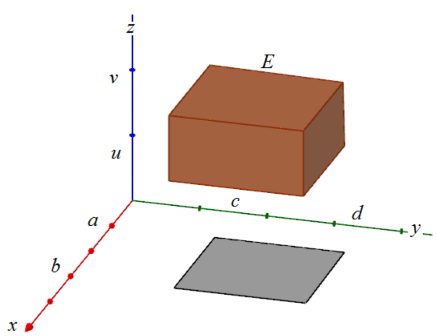
The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying region D and having density $\rho(x, y)$ are

$$\bar{x} = \frac{1}{m}\int\int_D x\rho(x, y)\, dA$$
$$\bar{y} = \frac{1}{m}\int\int_D y\rho(x, y)\, dA$$

Definition 60: Fubini's Theorem of Rectangles

Suppose $f(x, y, z)$ is continuous on the domain

$$E = \{(x, y, z) \mid a \leq b,\ c \leq y \leq d,\ u \leq z \leq v\} = [a, b] \times [c, d] \times [u, v]$$



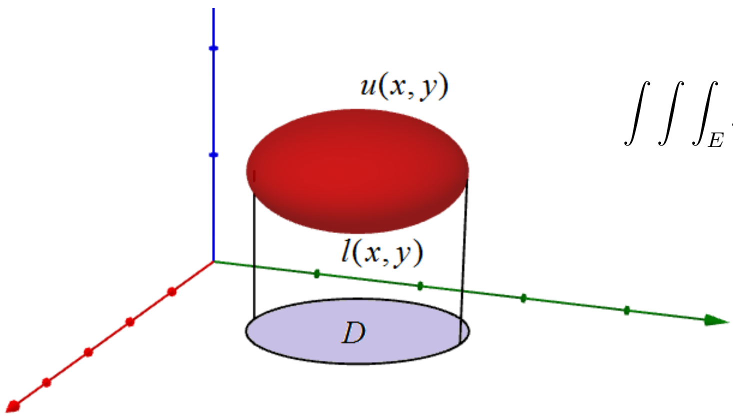
Fubini's Theorem states that this the integral $\int\int\int_E f(x, y, z)\, dV$ can be written six different ways.

$$\int_a^b \int_c^d \int_u^v f\ dz\ dy\ dx \qquad \int_c^d \int_a^b \int_u^v f\ dz\ dx\ dy \qquad \int_u^v \int_a^b \int_c^d f\ dy\ dx\ dz$$

$$\int_a^b \int_u^v \int_c^d f\ dy\ dz\ dx \qquad \int_c^d \int_u^v \int_a^b f\ dx\ dz\ dy \qquad \int_u^v \int_c^d \int_a^b f\ dx\ dy\ dz$$

Definition 61: z Simple - Type 1 Solid

A solid region E is said to be z Simple (Type 1) if z lies between two functions of x and y, $u(x, y)$ and $l(x, y)$ where D is the projection of E onto the xy plane.



$$\int\int\int_E f\ dV = \int\int_D \left[\int_{l(x,y)}^{u(x,y)} f\ dz\right]\ dA$$

Now that you have a visual on D, write D like you did in the previous section about general regions. Doing so will give you two possible integrals

$$\int_a^b \int_{b(x)}^{t(x)} \int_{l(x,y)}^{u(x,y)} f(x, y, z)\ dz\ dy\ dx$$

$$E = \{(x, y, z) \mid a \leq x \leq b,\ b(x) \leq y \leq t(x),\ l(x, y) \leq z \leq u(x, y)\}$$

OR

$$\int_c^d \int_{l(y)}^{r(y)} \int_{l(x,y)}^{u(x,y)} f(x, y, z)\ dz\ dx\ dy$$

$$E = \{(x, y, z) \mid c \leq y \leq d,\ l(y) \leq x \leq r(x),\ l(x, y) \leq z \leq u(x, y)\}$$

Definition 62: Volume of a Solid

Suppose $f(x, y, z) = 1$. Then the triple integral

$$\int\int\int_E 1\ dV = V(E)$$

where $V(E)$ is the volume of the solid E.

Definition 63: Convert Coordinates

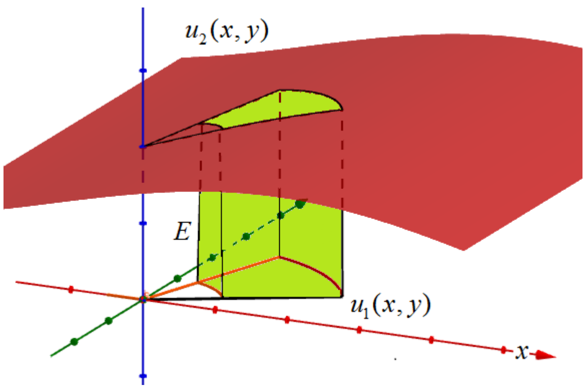
Cylindrical to Rectangular Coordinates

$$x = r\cos(\theta),\ y = r\sin(\theta),\ z = z$$

Rectangular to Cylindrical Coordinates

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

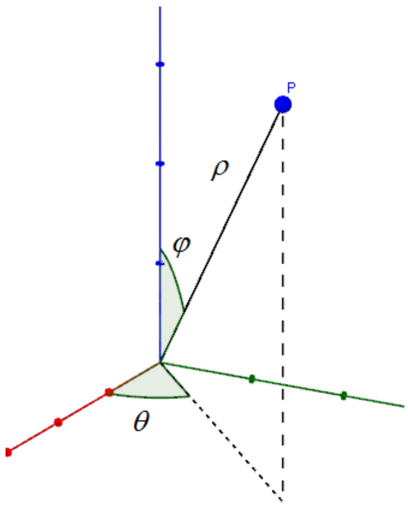
Definition 64: Triple Integrals in Cylindrical Coordinates



$$E = \begin{cases} \alpha \le \theta \le \beta \\ r_1(\theta) \le r \le r_2(\theta) \\ u_1(r\cos\theta, r\sin\theta) \le z \le u_2(r\cos\theta, r\sin\theta) \end{cases}$$

$$\int\int\int_E f(x,y,z)\ dV = \int_{\alpha}^{\beta}\int_{r_1(\theta)}^{r_2(\theta)}\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta)r\ dz\ dr\ d\theta$$

Definition 65: Spherical Coordinates



Convert to Rectangular Coordinates

$$x = \rho\cos(\theta)\sin(\phi)$$

$$y = \rho\sin(\theta)\sin(\phi)$$

$$z = \rho\cos(\phi)$$

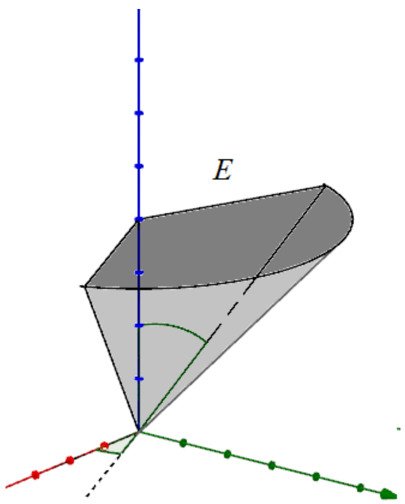
Convert to Spherical Coordinates

$$x^2 + y^2 + z^2 = \rho^2$$

$$\cos(\phi) = \frac{z}{\rho}$$

$$\cos(\theta) = \frac{x}{\rho\sin(\phi)}$$

Definition 66: Spherical Change of Coordinates



$$E = \begin{cases} a \le \rho \le b \\ c \le \theta \le d \\ \gamma \le \phi \le \delta \end{cases}$$

$$\int\int\int_E f(x, y, z)\, dV$$

$$= \int_a^b \int_c^d \int_\gamma^\delta f(\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \cdot \rho^2\sin(\phi)\, d\phi\, d\theta\, d\rho$$

Definition 67: Finding the Volume of E

Find the volume of E. Let $f(x, y, z) = 1$.

$$V = \int_a^b \int_c^d \int_\gamma^\delta 1 \cdot \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho$$

Definition 68: Jacobian

Suppose $x = g(u, v)$ and $y = h(u, v)$. Then the Jacobian of the transformation is given by

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Theorem 8: Transformation - Change of Variable

$$\int\int_R f(x, y)\, dA = \int\int_S f(g(u, v), h(u, v)) \cdot |J|\, dA$$

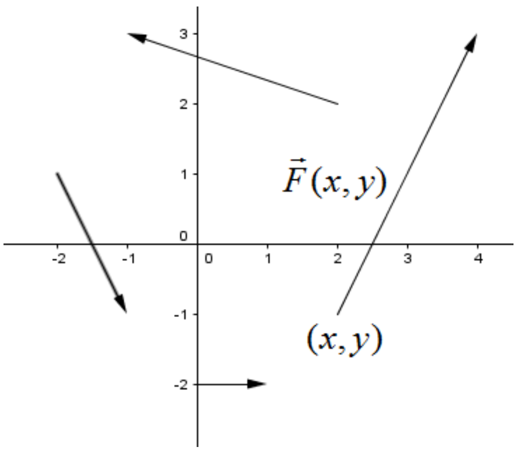
1. Sketch R
2. Use the Transformation $x = g(u, v)$ and $y = h(u, v)$ to transform the boundary lines of R to create S.

At this point S can either be described in rectangular coordinates or polar. It depends on what S looks like.

Chapter 16 - Vector Calculus

Definition 69: Vector Fields

Let D be a set in $\mathbf{R}^2$. A Vector Field on $\mathbf{R}^2$ is a function F that assigns each point (x, y) a two dimensional vector $F(x, y)$.



We write

$$F(x, y) = P(x, y)i + Q(x, y)j$$
$$= \langle P(x, y), Q(x, y)\rangle$$

Definition 70: Gradient Fields

Given a function $f(x, y)$ and its gradient $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$, a gradient vector field is the vector field using ∇f.

Definition 71: Line Integral of f along C

f is defined on a smooth curve C given by $r(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$.

$$\int_C f(x, y)\, dS = \int_a^b f\left(x(t), y(t)\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Definition 72

Suppose C_i is piecewise smooth with $C = C_1 + C_2 + ... + C_n$

$$\int_C f(x, y)\, dS = \int_{C_1} f(x, y)\, dS + \int_{C_2} f(x, y)\, dS + ... + \int_{C_n} f(x, y)\, dS$$

Definition 73: Line Integral with Respect to x or y

$$\int_C f(x,y)\ dx = \int_a^b f\left(x(t), y(t)\right)\ x'(t)\ dt$$

$$\int_C f(x,y)\ dy = \int_a^b f((x(t), y(t))\ y'(t)\ dt$$

$$\int_C P(x,y)\ dx + Q(x,y)\ dy = \int_C P(x,y)\ dx + \int_C Q(x,y)\ dy$$

Definition 74: General Line Integral of Vector Fields

Let $\mathbf{F}$ be a continuous vector field and C a smooth curve given by vector function $r(t)$, $a \le t \le b$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(r(t)) \cdot r'(t)\ dt$$

Definition 75: Fundamental Theorem of Line Integrals

C is a smooth curve given by vector function $r(t)$, $a \le t \le b$. Let f be a function such that $\nabla f = \langle f_x, f_y \rangle$ is continuous on C

$$\int_C \nabla f \cdot d\mathbf{r} = f(r(b)) - f(r(a))$$

Note: Line Integrals of conservative vector fields are independent of the path (as long as they have the same initial and terminal points).

Definition 76: Some Notes

1. $F(x, y)$ is usually defined by
$$F(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$
2. F is conservative if
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
3. If F is conservative then
$$F = \nabla f$$
and use the Fundamental Theorem of Line Integrals. To find f it must satisfy both
$$f = \int P(x, y)\, dx$$
$$f = \int Q(x, y)\, dy$$
4. If F is not conservative then you must use the General Line Integral of Vector Fields formula.
5. If F is a conservative vector field over a closed path C, then
$$\int_C F \cdot d\mathbf{r} = 0$$

Theorem 9: Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P(x, y)\, dx + Q(x, y)\, dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$$

The line integral can be converted into a double integral from chapter 15.

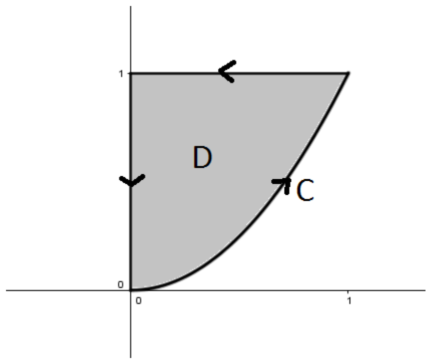
Example 1

Use Green's Theorem to evaluate $\displaystyle\int_C x^2y^2\, dx + xy\, dy$ where C is the arc of $y = x^2$ from $(0, 0)$ to $(1, 1)$, line segments from $(1, 1)$ to $(0, 1)$ and from $(0, 1)$ to $(0, 0)$.

1. $P = x^2y^2$ and $\dfrac{\partial P}{\partial y} = 2x^2y$

2. $Q = xy$ and $\dfrac{\partial Q}{\partial x} = y$

3. Sketch D



$$D = \left\{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le 1\right\}$$

$$\begin{aligned} \int_C x^2y^2\,dx + xy\,dy &= \int\int_D y - 2x^2y\,dA \\ &= \int_0^1 \int_{x^2}^1 y - 2x^2y\,dy\,dx \end{aligned}$$