Math 232

Calculus III

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16.4 Green's Theorem

Theorem 1

$\int_C \mathbf{F} \cdot dr$ is independent of path in D if and only if

$$\int_C \mathbf{F} \cdot dr = 0$$

for every closed path C in D.

In the last section we found out that if $\mathbf{F}$ was a conservative vector field then we had a nice way to integrate it over a curve. As long as the initial and terminal points were the same, the integral did not depend on the path chosen. But what happens if F is not conservative? In 16.2 we needed to evaluate the integral the hard way

$$\int_C \mathbf{F} \cdot dr = \int_a^b \mathbf{F}(r(t)) \cdot r'(t)\, dt$$

Fortunately, in this section we can evaluate the integral $\int_C \mathbf{F} \cdot dr$ easily even if $\mathbf{F}$ is not conservative. But there are conditions on the domain and path.

Theorem 2: Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

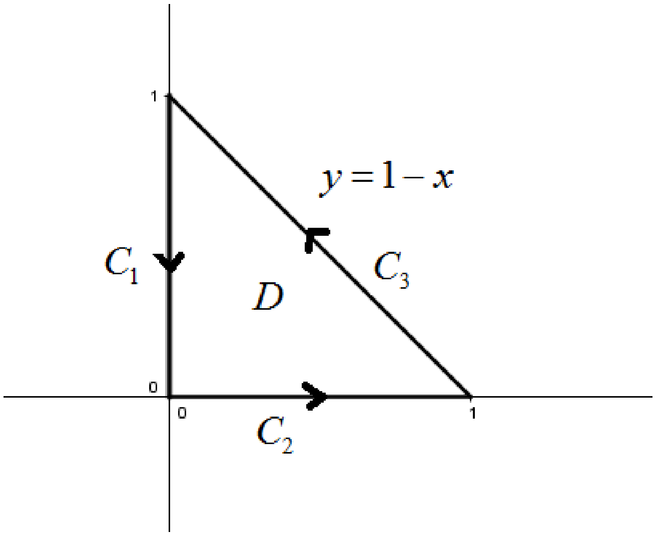
$$\int_C P(x,y)\, dx + Q(x,y)\, dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\, dA$$

The line integral can be converted into a double integral from chapter 15.

Example 1

Evaluate $\int_C x^4\, dx + xy\, dy$ where C is the triangle formed by the points $(0,0)$, $(1,0)$ and $(1,0)$.

1. Let's look at the path C and the region D formed by the triangle.



2. Let's solve this using the direct method from 16.2 by integrating over the curves C_1, C_2, and C_3 separately.

   (a) Over C_1

      i. $r(t) = \langle x(t), y(t)\rangle = \langle 0, 1-t\rangle$, $0 \le x \le 1$

      ii. $r'(t) = \langle x'(t), y'(t)\rangle = \langle 0, -1\rangle$

      iii. Formulas Needed:

$$\int_C P(x,y)\, dx = \int_a^b P(x(t), y(t))x'(t)\, dt$$

$$\int_C Q(x,y)\, dy = \int_a^b Q(x(t), y(t))y'(t)\, dt$$

      iv. Formula: $\int_{C_1} \mathbf{F} \cdot dr = \int_a^b \mathbf{F(r(t))} \cdot r'(t)\, dt = \int_a^b P(x,y)\, dx + Q(x,y)\, dy$

$$\begin{aligned}\int_{C_1} x^4\,dx + xy\,dy &= \int_0^1 (0)^4(0\,dt) + 0(1-t)(-1\,dt)\\ &= \int_0^1 0\,dt\\ &= 0\end{aligned}$$

(b) Over C_2

i. $r(t) = \langle x(t), y(t)\rangle = \langle t, 0\rangle$, $0 \le t \le 1$

ii. $r'(t) = \langle x'(t), y'(t)\rangle = \langle 1, 0\rangle$

$$\begin{aligned}\int_{C_2} x^4\,dx + xy\,dy &= \int_0^1 (t)^4(1\,dt) + (t)(0)(0\,dt)\\ &= \int_0^1 t^4\,dt\\ &= \frac{1}{5}t^5\Big|_0^1\\ &= \frac{1}{5}\end{aligned}$$

(c) Over C_3

i. $r(t) = \langle x(t), y(t)\rangle = \langle 1-t, t\rangle$

ii. $r'(t) = \langle x'(t), y'(t)\rangle = \langle -1, 1\rangle$

$$\begin{aligned}\int_{C_3} x^4\,dx + xy\,dy &= \int_0^1 (1-t)^4(-1\,dt) + (1-t)t(1\,dt)\\ &= \int_0^1 -(1-t)^4 + t - t^2\,dt\\ &= \frac{1}{5}(1-t)^5 + \frac{1}{2}t^2 - \frac{1}{3}t^3\Big|_0^1\\ &= -\frac{1}{30}\end{aligned}$$

(d) $\displaystyle\int_{C_1+C_2+C_3} x^4\,dx + xy\,dy = 0 + \frac{1}{5} - \frac{1}{30} = \frac{1}{6}$

3. Let's try with Green's Theorem

   (a) Let $P = x^4$ and $\dfrac{\partial P}{\partial y} = 0$

   (b) Let $Q = xy$ and $\dfrac{\partial Q}{\partial x} = y$

   (c) $D = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x\}$

$$\begin{aligned}
\int_C x^4\,dx + xy\,dy &= \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA \\
&= \int\int_D y\,dA \\
&= \int_0^1 \int_0^{1-x} y\,dy\,dx \\
&= \int_0^1 \left[\frac{1}{2}y^2\Big|_0^{1-x}\right]\,dx \\
&= \int_0^1 \frac{1}{2}(1-x)^2\,dx \\
&= -\frac{1}{6}(1-x)^3\Big|_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

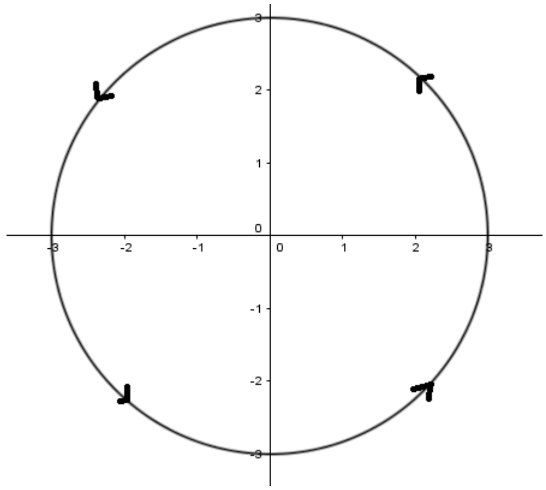
Example 2

Use Green's Theorem to evaluate $\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4 + 1})\,dy$, where C is the circle $x^2 + y^2 = 9.1$

1. Let $P = 3y - e^{\sin x}$ and $\dfrac{\partial P}{\partial y} = 3$

2. Let $Q = 7x + \sqrt{y^4 + 1}$ and $\dfrac{\partial Q}{\partial x} = 7$

3. Sketch D



$$D = \{(r, \theta) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$$

4. The region D is best described in polar. So we need to change the integral to polar using

$$x = r\cos\theta$$

$$y = r\sin\theta$$

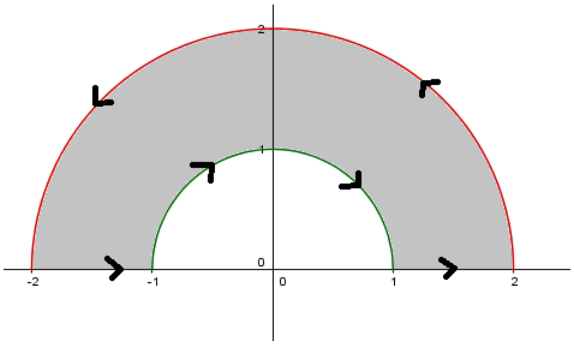
$$x^2 + y^2 = r^2$$

$$\begin{aligned}
\int_C (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4+1})\, dy &= \int\int_D (7-3)\, dA \\
&= \int\int_D 4\, dA \\
&= \int_0^{2\pi}\int_0^3 4r\, dr\, d\theta \\
&= \int_0^{2\pi} \left[2r^2\Big|_0^3 \right]\, d\theta \\
&= \int_0^{2\pi} 18\, d\theta \\
&= 18\theta\Big|_0^{2\pi} \\
&= 36\pi
\end{aligned}$$

Example 3

Use Green's Theorem to evaluate $\int_C y^2\, dx + 3xy\, dy$ where C is the boundary of the semicircular region D is the upper half plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

1. $P = y^2$ and $\dfrac{\partial P}{\partial y} = 2y$
2. $Q = 3xy$ and $\dfrac{\partial Q}{\partial x} = 3y$
3. Sketch D



$$D = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$$

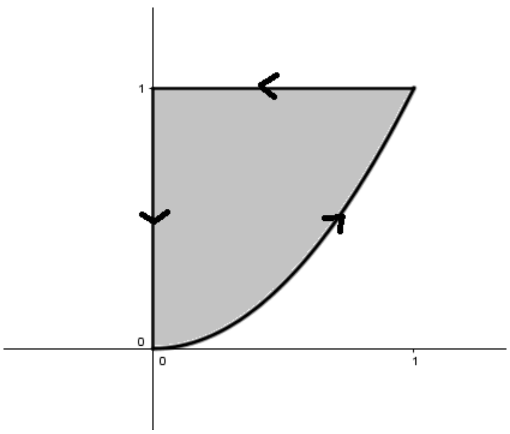
The region is best described in polar.

$$\begin{aligned}
\int_C y^2\, dx + 3xy\, dy &= \int\int_D (3y - 2y)\, dA \\
&= \int\int_D y\, dA \\
&= \int_0^{\pi}\int_1^2 r\sin(\theta)\cdot r\, dr\, d\theta \\
&= \int_0^{\pi} \sin\theta\, d\theta \cdot \int_1^2 r^2\, dr \\
&= -\cos\theta\big|_0^{\pi} + \frac{1}{3}r^3\Big|_1^2 \\
&= 2\cdot\frac{7}{3} \\
&= \frac{14}{3}
\end{aligned}$$

Example 4

Use Green's Theorem to evaluate $\int_C x^2y^2\ dx + xy\ dy$ where C is the arc of $y = x^2$ from $(0,0)$ to $(1,1)$, line segments from $(1,1)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$.

1. $P = x^2y^2$ and $\dfrac{\partial P}{\partial y} = 2x^2y$
2. $Q = xy$ and $\dfrac{\partial Q}{\partial x} = y$
3. Sketch D



$D = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le 1\}$

$$\begin{aligned}
\int_C x^2y^2\ dx + xy\ dy &= \iint_D y - 2x^2y\ dA \\
&= \int_0^1 \int_{x^2}^1 y - 2x^2y\ dy\ dx \\
&= \int_0^1 \left[\frac{1}{2}y^2 - x^2y^2 \Big|_{x^2}^1 \right] dx \\
&= \int_0^1 x^6 - \frac{1}{2}x^4 - x^2 + \frac{1}{2}\ dx \\
&= \frac{1}{7}x^7 - \frac{1}{10}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x \Big|_0^1 \\
&= \frac{22}{105}
\end{aligned}$$