16.3 The Fundamental Theorem for Line Integrals

When you first learned about the Fundamental Theorem of Calculus, you learned that

\[ \int_{a}^{b} F'(x) \, dx = F(b) - F(a) \]

where \( F' \) is continuous and the derivative of \( F \).

For a function of two or more variables we used the gradient of \( f \), \( \nabla f \), to represent the derivative of \( f \). This leads us into the Fundamental Theorem for Line Integrals.

**Definition 1: The Fundamental Theorem for Line Integrals**

Assume \( C \) is a smooth curve given by the vector function \( r(t) \), \( a \leq t \leq b \). Let \( f \) be a differentiable function such that \( \nabla f \) is continuous on \( C \). Then

\[ \int_{C} \nabla f \cdot dr = f(r(b)) - f(r(a)) \]

Note: Line Integrals of conservative vector fields are independent of the path (as long as they have the same initial and terminal points).

**Theorem 1: Conservative Vector Fields**

Let \( F = Pi + Qj \) be a vector field. Then \( F \) is conservative if \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \) on \( D \). Another way of defining a conservative vector field is to say there exists a function \( f \) such that \( F = \nabla f \).

\[
\begin{align*}
P(x, y) &= f_x = \frac{\partial f}{\partial x} \\
Q(x, y) &= f_y = \frac{\partial f}{\partial y}
\end{align*}
\]
A curve \( C \) is \textbf{CLOSED} if its terminal point is the same as its initial point.

### Example 1

Determine whether or not the vector field \( F(x, y) = (x - y)i + (x - 2)j \) is conservative.

Let \( P(x, y) = x - y \) and \( Q(x, y) = x - 2 \). Then

\[
\frac{\partial P}{\partial y} = -1 \\
\frac{\partial Q}{\partial x} = 1
\]

Since \( \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \), \( F \) is not conservative.

### Example 2

Determine whether or not the vector field \( F(x, y) = (3 + 2xy)i + (x^2 - 3y^2)j \) is conservative.

Let \( P(x, y) = 3 + 2xy \) and \( Q(x, y) = x^2 - 3y^2 \). Then

\[
\frac{\partial P}{\partial y} = 2x \\
\frac{\partial Q}{\partial x} = 2x
\]

Since \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \), \( F \) is conservative.

### Example 3

Let \( F(x, y) = (3 + 2xy)i + (x^2 - 3y^2)j \), find a function \( f \) such that \( F = \nabla f \). Then evaluate \( \int_C F \cdot dr \), where \( C \) is given by

\[
r(t) = e^t \sin(t)i + e^t \cos(t)j, \quad 0 \leq t \leq \pi
\]

1. Let’s start by finding \( f \) such that \( F = \nabla f = \langle f_x, f_y \rangle \).

Matching up components, we get

\[
P(x, y) = \frac{\partial f}{\partial x} = 3 + 2xy
\]
\[
f = \int \frac{\partial f}{\partial x} \, dx = 3x + x^2y + g(y)
\]

Think of \(g(y)\) as the \(+C\). The reason we have \(g(y)\) and not \(+C\) is I integrated with respect to \(x\) only. That means \(y\)'s are considered constant. Now we only need to find \(g(y)\).

We now need to find \(g(y)\). To find \(g(y)\) recognize

\[
Q(x,y) = \frac{\partial f}{\partial y} = x^2 - 3y^2 = x^2 + g'(y)
\]

\[
-3y^2 = g'(y)
\]

To find \(g(y)\):

\[
g(y) = \int g'(y) \, dy = -y^3 + K
\]

This gives us \(f(x,y) = 3x + x^2y - y^3 + K\)

2. Now evaluate \(\int_C \mathbf{F} \cdot d\mathbf{r}\) where \(r(t) = \langle e^t \sin(t), e^t \cos(t) \rangle\), \(0 \leq t \leq \pi\).

(a) To use the Fundamental Theorem for Line Integrals we need to the initial and terminal points:

\[
r(0) = \langle e^0 \sin(0), e^0 \cos(0) \rangle = \langle 0, 1 \rangle
\]

\[
r(\pi) = \langle e^\pi \sin(\pi), e^\pi \cos(\pi) \rangle = \langle 0, -e^\pi \rangle
\]

(b) Now use the Fundamental Theorem
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(r(\pi)) - f(r(0)) = f(0, -e^\pi) - f(0, 1) = \left[3(0) + (0)^2(e^{-\pi}) - (-e^\pi)^3\right] - \left[3(0) + (0)^2(1) - (1)^3\right] = e^{3\pi} + 1
\]

Example 4

Let \( F(x, y) = y^2e^{xy} \mathbf{i} + (1 + xy)e^{xy} \mathbf{j} \)

1. Show \( F \) is conservative.

2. Find \( f \) such that \( F = \nabla f \).

3. Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C : r(t) = (\cos(t), 2\sin(t)) \), \( 0 \leq t \leq \pi/2 \)

1. Show \( F \) is conservative.

Note that \( P(x, y) = y^2e^{xy} \) and \( Q(x, y) = (1 + xy)e^{xy} \).

\[
\frac{\partial P}{\partial y} = xy^2e^{xy} + ye^{xy}
\]

\[
\frac{\partial Q}{\partial x} = y(1 + xy)e^{xy} + ye^{xy} = 2ye^{xy} + xy^2e^{xy}
\]

Since \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \), \( F \) is conservative.

2. Find \( f \) such that \( F = \nabla f \)

\[
f = \int P \, dx = \int y^2e^{xy} \, dx = ye^{xy} + g(y)
\]

\[
\frac{\partial f}{\partial y} = xye^{xy} + e^{xy} + g'(y)
\]
Since \( \frac{\partial f}{\partial y} = Q(x, y) \), we can set them equal.

\[
xy^y + e^{xy} + g'(y) = (1 + xy)e^{xy}
\]

Matching these, we get \( g'(y) = 0 \), and so \( g(y) = K \).

\[
f(x, y) = ye^{xy} + K
\]

3. Evaluate \( \int_C F \cdot dr \).

(a) To use the Theorem, we need \( r(\pi/2) \) and \( r(0) \)

\[
r(\pi/2) = \langle \cos(\pi/2), 2\sin(\pi/2) \rangle = \langle 0, 2 \rangle
\]

\[
r(0) = \langle \cos(0), 2\sin(0) \rangle = \langle 1, 0 \rangle
\]

(b) Use the Fundamental Theorem for Line Integrals

\[
\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(\pi/2)) - f(r(0))
\]

\[
= f(0, 2) - f(1, 0)
\]

\[
= 2e^0 - 0e^0
\]

\[
= 2
\]

**Definition 2: Some Notes**

Normally \( F \) is defined as

\[
F(x, y) = P(x, y)i + Q(x, y)j
\]

But know that

\[
\int_C F \cdot dr = \int_C P(x, y) \, dx + Q(x, y) \, dy
\]