Math 232

Calculus III

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16.3 The Fundamental Theorem for Line Integrals

When you first learned about the Fundamental Theorem of Calculus, you learned that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where F' is continuous and the derivative of F.

For a function of two or more variables we used the gradient of f, ∇f, to represent the derivative of f. This leads us into the Fundamental Theorem for Line Integrals.

Definition 1: The Fundemental Theorem for Line Integrals

Assume C is a smooth curve given by the vector function $r(t)$, $a \leq t \leq b$. Let f be a differentiable function such that ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(r(b)) - f(r(a))$$

Note: Line Integrals of conservative vector fields are independent of the path (as long as they have the same initial and terminal points).

Theorem 1: Conservative Vector Frields

Let $F = P\mathbf{i} + Q\mathbf{j}$ be a vector field. Then F is conservative if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on D. Another way of defining a conservative vector field is to say there exists a function f such that $F = \nabla f$.

$$P(x,y) = f_x = \frac{\partial f}{\partial x}$$
$$Q(x,y) = f_y = \frac{\partial f}{\partial y}$$

A curve C is **CLOSED** if its terminal point is the same as its initial point.

Example 1

Determine whether or not the vector field $F(x,y) = (x-y)\mathbf{i} + (x-2)\mathbf{j}$ is conservative.

Let $P(x,y) = x - y$ and $Q(x,y) = x - 2$. Then

$$\frac{\partial P}{\partial y} = -1$$
$$\frac{\partial Q}{\partial x} = 1$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, F is not conservative.

Example 2

Determine whether or not the vector field $F(x,y) = (3+2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is conservative.

Let $P(x,y) = 3 + 2xy$ and $Q(x,y) = x^2 - 3y^2$. Then

$$\frac{\partial P}{\partial y} = 2x$$
$$\frac{\partial Q}{\partial x} = 2x$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, F is conservative.

Example 3

Let $F(x,y) = (3+2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, find a function f such that $F = \nabla f$. Then evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by

$$r(t) = e^t \sin(t)\mathbf{i} + e^t \cos(t)\mathbf{j},\ 0 \le t \le \pi$$

1. Let's start by finding f such that $F = \nabla f = \langle f_x, f_y \rangle$.

   Matching up components, we get

$$P(x,y) = \frac{\partial f}{\partial x} = 3 + 2xy$$

$$f = \int \frac{\partial f}{\partial x}\, dx = 3x + x^2 y + g(y)$$

Think of $g(y)$ as the $+C$. The reason we have $g(y)$ and not $+C$ is I integrated with respect to x only. That means y's are considered constant. Now we only need to find $g(y)$.

We now need to find $g(y)$. To find $g(y)$ recognize

$$Q(x,y) = \frac{\partial f}{\partial y}$$

$$x^2 - 3y^2 = x^2 + g'(y)$$

$$-3y^2 = g'(y)$$

To find $g(y)$:

$$g(y) = \int g'(y)\, dy = -y^3 + K$$

This gives us $f(x,y) = 3x + x^2 y - y^3 + K$

2. Now evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $r(t) = \langle e^t \sin(t), e^t \cos(t) \rangle$, $0 \leq t \leq \pi$.

   (a) To use the Fundamental Theorem for Line Integrals we need to the initial and terminal points:

   $$r(0) = \langle e^0 \sin(0), e^0 \cos(0) \rangle = \langle 0, 1 \rangle$$

   $$r(\pi) = \langle e^\pi \sin(\pi), e^\pi \cos(\pi) \rangle = \langle 0, -e^\pi$$

   (b) Now use the Fundamental Theorem

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r}\\ &= f(r(\pi)) - f(r(0))\\ &= f(0,-e^{\pi}) - f(0,1)\\ &= \left[3(0)+(0)^2(e^{-\pi}) - (-e^{\pi})^3\right] - \left[3(0)+(0)^2(1)-(1)^3\right]\\ &= e^{3\pi}+1\end{aligned}$$

Example 4

Let $F(x,y) = y^2e^{xy}\mathbf{i} + (1+xy)e^{xy}\mathbf{j}$

1. Show F is conservative.
2. Find f such that $F = \nabla f$.
3. Evaluate $\int_C F\cdot dr$ where $C:\ r(t) = \langle \cos(t), 2\sin(t)\rangle,\ 0 \le t \le \pi/2$

1. Show F is conservative.

   Note that $P(x,y) = y^2e^{xy}$ and $Q(x,y) = (1+xy)e^{xy}$.

$$\frac{\partial P}{\partial y} = xy^2e^{xy} + 2ye^{xy}$$

$$\frac{\partial Q}{\partial x} = y(1+xy)e^{xy} + ye^{xy} = 2ye^{xy} + xy^2e^{xy}$$

   Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, F is conservative.

2. Find f such that $F = \nabla f$

$$f = \int P\,dx = \int y^2e^{xy}\,dx = ye^{xy} + g(y)$$

$$\frac{\partial f}{\partial y} = xye^{xy} + e^{xy} + g'(y)$$

Since $\dfrac{\partial f}{\partial y} = Q(x,y)$, we can set them equal.

$$xy^{xy} + e^{xy} + g'(y) = (1+xy)e^{xy}$$

Matching these, we get $g'(y) = 0$, and so $g(y) = K$.

$$f(x,y) = ye^{xy} + K$$

3. Evaluate $\displaystyle\int_C F \cdot dr$.

   (a) To use the Theorem, we need $r(\pi/2)$ and $r(0)$

   $$r(\pi/2) = \langle \cos(\pi/2), 2\sin(\pi/2) \rangle = \langle 0, 2 \rangle$$

   $$r(0) = \langle \cos(0), 2\sin(0) \rangle = \langle 1, 0 \rangle$$

   (b) Use the Fundamental Theorem for Line Integrals

   $$\begin{aligned} \int_C F \cdot dr = \int_C \nabla f \cdot dr &= f(r(\pi/2)) - f(r(0)) \\ &= f(0,2) - f(1,0) \\ &= 2e^0 - 0e^0 \\ &= 2 \end{aligned}$$

Definition 2: Some Notes

Normally F is defined as

$$F(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

But know that

$$\int_C F \cdot dr = \int_C P(x,y)\,dx + Q(x,y)\,dy$$