MATH 232

CALCULUS III

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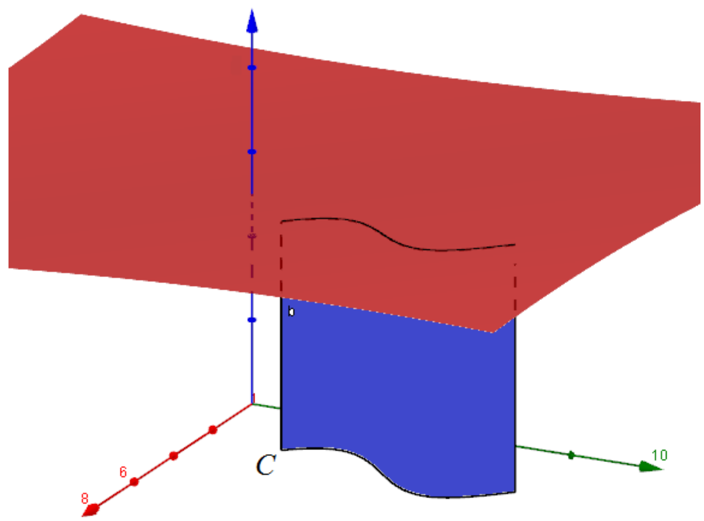
16.2 Line Integrals

Instead of integrating over a region D like we did in Chapter 15, we integrate over a curve C. We call these line integrals. The resulting integral will give us the area of the 2 dimensional shape (indicated by the blue region) under the function $f(x, y)$.



We need a way to represent the curve C. As it is a line, we can represent it as parametric equations $x = x(t)$ and $y = y(t)$ from $t = a$ to $t = b$.

Definition 1: Line Integral of f along C

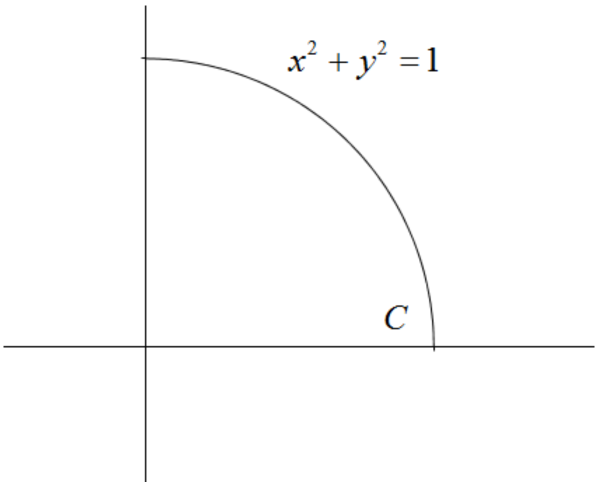
If f is defined on a smooth curve C given by the parametric equations $x = x(t)$ and $y = y(t)$ $a \le t \le b$, then the line integral of f along C is

$$\int_C f(x, y)\, dS = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 1

Evaluate $\int_C x + 2y \; dS$, where C is the portion of the circle $x^2 + y^2 = 1$ in the first quadrant.

1. Let's take a look at the curve C we are integrating over.



According to the line integral definition, we need to write C in parametric equations. Note that parametric equations are not unique. In this example, we can write the curve two different ways.

   (a) $x = \cos(t)$, $y = \sin(t)$ on $0 \le t \le \pi/2$.

   (b) If you solve for y, $y = \sqrt{1 - x^2}$, we can write the curve as

$$x = t, \; y = \sqrt{1 - t^2}, \; 0 \le t \le 1$$

2. Evaluating the integral using $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le \pi/2$.

   (a) $\dfrac{dx}{dt} = -\sin(t)$

   (b) $\dfrac{dy}{dt} = \cos(t)$

   (c) Substitute

$$\begin{aligned}
\int_C x + 2y\, dS &= \int_0^{\pi/2} (\cos(t) + 2\sin(t))\sqrt{(-\sin(t)^2 + (\cos(t))^2}\, dt \\
&= \int_0^{\pi/2} (\cos t + 2\sin t)\sqrt{1}\, dt \\
&= \int_0^{\pi/2} \cos t + 2\sin t\, dt \\
&= \sin t - 2\cos t\big|_0^{\pi/2} \\
&= [\sin(\pi/2) - 2\cos(\pi/2)] - [\sin(0) - 2\cos(0)] \\
&= 3
\end{aligned}$$

3. Evaluating the integral using $x = t$, $y = \sqrt{1-t^2}$, $0 \le t \le 1$.

   (a) $\dfrac{dx}{dt} = 1$

   (b) $\dfrac{dy}{dt} = \dfrac{-t}{\sqrt{1-t^2}}$

   (c) Substitute

$$\begin{aligned}
\int_C x + 2y\, dS &= \int_0^1 \left(t + 2\sqrt{1-t^2}\right)\sqrt{(1)^2 + \left(\frac{-t}{\sqrt{1-t^2}}\right)^2}\, dt \\
&= \int_0^1 \left(t + 2\sqrt{1-t^2}\right)\sqrt{1 + \frac{t^2}{1-t^2}}\, dt \\
&= \int_0^1 \left(t + 2\sqrt{1-t^2}\right)\sqrt{\frac{1}{1-t^2}}\, dt \\
&= \int_0^1 \left(t + 2\sqrt{1-t^2}\right)\frac{1}{\sqrt{1-t^2}}\, dt \\
&= \int_0^1 \frac{t}{\sqrt{1-t^2}} + 2\, dt \\
&= \int_0^1 \frac{t}{\sqrt{1-t^2}}\, dt + \int_0^1 2\, dt
\end{aligned}$$

   (d) $\displaystyle\int_0^1 \frac{t}{\sqrt{1-t^2}}\, dt$

      i. Let $u = 1 - t^2$

ii. $du = -2t\ dt \Rightarrow -\frac{1}{2}\ du = t\ dt$

iii. If $t = 1$, $u = 1 - 1^2 = 0$

iv. If $t = 0$, $u = 1 - 0^2 = 1$

v. Substitute

$$\begin{aligned}\int_0^1 \frac{t}{\sqrt{1-t^2}}\ dt &= \int_1^0 -\frac{1}{2}u^{-1/2}\ du \\ &= -u^{1/2}\Big|_1^0 \\ &= -0^{1/2} + 1^{1/2} \\ &= 1\end{aligned}$$

(e) $\int_0^1 2\ dt = 2t|_0^1 = 2$

4. Final Answer: $\int_C x + 2y\ dS = \int_0^1 \left(t + 2\sqrt{1-t^2}\right)\sqrt{(1)^2 + \left(\frac{-t}{\sqrt{1-t^2}}\right)^2}\ dt = 2 + 1 = 3$

Definition 2: Piecewise-Smooth Curve

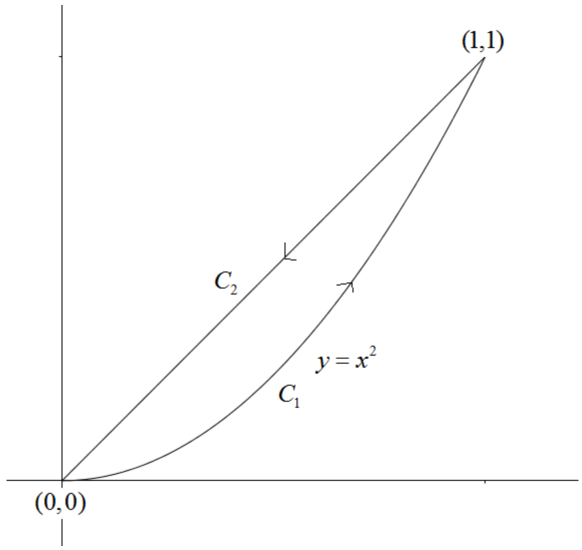
Suppose C is piecewise-smooth. We can say $C = C_1 + C_2 + ... + C_n$, where each C_n is smooth. Then

$$\int_C f(x,y)\ dS = \int_{C_1} f(x,y)\ dS + \int_{C_2} f(x,y)\ dS + ... + \int_{C_n} f(x,y)\ dS$$

Example 2

valuate $\int x + \sqrt{y}\ dS$ where C is the parabola $y = x^2$ connecting $(0,0)$ to $(1,1)$ and the line segment connecting $(1,1)$ to $(0,0)$.

1. Here's an image of C



2. Let's write out parametric equations for both C_1 and C_2.

$$C_1 : \ x = t, \ y = t^2, \ 0 \le t \le 1$$

$$C_2 : \ x = 1 - t, \ y = 1 - t, \ 0 \le t \le 1$$

3. Let's integrate over C_1

$$\begin{aligned}
\int_{C_1} x + \sqrt{y}\, dS &= \int_0^1 \left(t + \sqrt{t^2}\right) \sqrt{(1)^2 + (2t)^2}\, dt \\
&= \int_0^1 2t\sqrt{1 + 4t^2}\, dt \\
&\quad \text{Let } u = 1 + 4t^2, \ \frac{1}{8}\, du = t\, dt \\
&= \int_1^5 \frac{1}{4} u^{1/2}\, du \\
&= \frac{1}{6} u^{3/2} \Big|_1^5 \\
&= \frac{1}{6} 5^{3/2} - \frac{1}{6}
\end{aligned}$$

4. Let's integrate over C_2

$$\begin{aligned}
\int_{C_2} x + \sqrt{y}\, dS &= \int_0^1 \left(1 - t + \sqrt{1-t}\right) \sqrt{(-1)^2 + (-1)^2}\, dt \\
&= \sqrt{2} \int_0^1 1 - t + \sqrt{1-t}\, dt \\
&= \sqrt{2} \left[t - \frac{1}{2}t^2 - \frac{2}{3}(1-t)^{3/2} \Big|_0^1 \right] \\
&= \sqrt{2} \left[1 - \frac{1}{2} + 0 + 0 + \frac{2}{3} \right] \\
&= 7\sqrt{2}/6
\end{aligned}$$

5. Final Answer: $\displaystyle\int_C x + \sqrt{y}\, dS = \frac{1}{6}5^{3/2} - \frac{1}{6} + 7\sqrt{2}/6 = \frac{1}{6}\left(5^{3/2} - 1 - 7\sqrt{2}\right)$

Definition 3: Line Integral with Respect to x or y

$$\int_C f(x,y)\, dx = \int_a^b f\left(x(t), y(t)\right)\; x'(t)\, dt$$

$$\int_C f(x,y)\, dy = \int_a^b f\left(x(t), y(t)\right)\; y'(t)\, dt$$

Sometimes line integrals with respect to x and y occur together. When this happens we write it as

$$\int_C P(x,y)\, dx + \int_C Q(x,y)\, dy = \int_C P(x,y)\, dx + Q(x,y)\, dy$$

Example 3

Evaluate $\displaystyle\int_C y^2\, dx + x\, dy$, where

1. C is the line segment from $(-5,-3)$ to $(0,2)$
2. C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

NOTE: These are two separate problems.

1. C is the line segment from $(-5,-3)$ to $(0,2)$

$$x = -5 + 5t,\ y = -3 + 5t,\ 0 \le t \le 1$$

$$\begin{aligned}
\int_C y^2\,dx + x\,dy &= \int_0^1 (-3+5t)^2 \cdot (5)\,dt + (-5+5t)\cdot(5)\,dt \\
&= \int_0^1 5(9 - 30t + 25t^2) + 25t - 25\,dt \\
&= \int_0^1 25t^2 - 125t - 20\,dt \\
&= \frac{125}{3}t^3 - \frac{125}{2}t^2 + 20t\Big|_0^1 \\
&= \frac{125}{3} - \frac{125}{2} + 20 \\
&= -\frac{5}{6}
\end{aligned}$$

2. C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$

$$x = 4 - t^2,\ y = t,\ -3 \le t \le 2$$

$$\begin{aligned}
\int_C y^2\,dx + x\,dy &= \int_{-3}^2 t^2(-2t)\,dt + (4-t^2)(1)\,dt \\
&= \int_{-3}^2 -2t^3 - t^2 + 4\,dt \\
&= -\frac{1}{2}t^4 - \frac{1}{3}t^3 + 4t\Big|_{-3}^2 \\
&= -\frac{8}{3} + 43.5 \\
&\approx 40.833
\end{aligned}$$

Definition 4: Line Integrals of Vector Fields

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $r(t)$, $a \le t \le b$. Then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(r(t))\cdot r'(t)\,dt$$

Example 4

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ and C is given by

1. $C: \ r(t) = \cos(t)i + \sin(t)j$, $0 \le t \le \pi/2$
2. $C_2: \ r(t) = (1-t)i + tj$, $0 \le t \le 1$

Note that these two paths both take you from the initial point $(1, 0)$ to the terminal point $(0, 1)$.

1. Path $C: \ r(t) = \cos(t)i + \sin(t)j$, $0 \le t \le \pi/2$.
   (a) $F(x, y) = \langle x^2, -xy \rangle$
   (b) $r(t) = \langle \cos(t), \sin(t) \rangle$.
   (c) $F(r(t)) = \langle \cos^2(t), -\cos(t)\sin(t) \rangle$
   (d) $r'(t) = \langle -\sin(t), \cos(t) \rangle$

$$\begin{aligned}
\int_C \mathbf{F} \cdot dr &= \int_0^{\pi/2} \langle \cos^2(t), -\cos(t)\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \ dt \\
&= \int_0^{\pi/2} -\cos^2(t)\sin(t) - \cos^2(t)\sin(t) \ dt \\
&= \int_0^{\pi/2} -2\cos^2(t)\sin(t) \ dt \\
&\quad \text{Let } u = \cos(t), \ du = -\sin(t) \ dt \\
&= \int_1^0 2u^2 \ du \\
&= \frac{2}{3}u^3 \Big|_1^0 \\
&= 0 - \frac{2}{3} \\
&= -\frac{2}{3}
\end{aligned}$$

2. Path $C_2: \ r(t) = (1-t)i + tj$, $0 \le t \le 1$.
   (a) $F(x, y) = \langle x^2, -xy \rangle$

(b) $r(t) = \langle 1-t, t\rangle$

(c) $F(r(t)) = \langle (1-t)^2, -(1-t)t\rangle = \langle 1-2t+t^2, t^2-t\rangle$

(d) $r'(t) = \langle -1, 1\rangle$

$$\begin{aligned}
\int_{C_2} \mathbf{F}\cdot dr &= \int_0^1 \langle 1-2t+t^2, t^2-t\rangle \cdot \langle -1,1\rangle \, dt \\
&= \int_0^1 -1+2t-t^2+t^2-t \, dt \\
&= \int_0^1 t-1 \, dt \\
&= \frac{1}{2}t^2 - t\Big|_0^1 \\
&= \frac{1}{2} - 1 - 0 + 0 \\
&= -\frac{1}{2}
\end{aligned}$$

Note how the integral of the vector field $F(x, y)$ over two different paths gets you two different values. In the next section we find out that it is possible that you will get the same value NO MATTER WHICH PATH YOU TAKE. But the vector field function $F(x, y)$ must have a special property for this to work.