15.9 Change in Variables in Multiple Integrals

In one variable calculus a $u$ substitution would have looked like this:

$$\int_a^b f(x) \, dx = \int_c^d f(g(u)) \, g'(u) \, du$$

where $x = g(u)$, $dx = g'(u) \, du$, $a = g(c)$, and $b = g(d)$.

Now we consider a change of variables that is given by a transformation $(x, y) = T(u, v)$ where $x = g(u, v)$ and $y = h(u, v)$.

Suppose you have a bounded region in $(x, y)$ called $R$. A change in variables from $(x, y)$ to $T(u, v)$ maps the region $R$ to a new region $S$ that’s graphed on the $uv$ plane. In one variable calculus this would be equivalent to interval bounds.

**Example 1**

Determine the new region that we get by applying the transformation to $R$.

1. $R$ is the ellipse $x^2 + \frac{y^2}{36} = 1$ with $x = \frac{u}{2}$ and $y = 3v$.

2. $R$ is the region bounded by $y = -x + 4$, $y = x + 1$, and $y = \frac{x}{3} - \frac{4}{3}$ with $x = \frac{u + v}{2}$ and $y = \frac{u - v}{2}$.

1. Take one of the equations that involve $x$, $y$, or both. In this case there is only one function $x^2 + \frac{y^2}{36} = 1$. Plug in $x = \frac{u}{2}$ and $y = 3v$.

\[
\left(\frac{u}{2}\right)^2 + \left(\frac{3v}{36}\right)^2 = 1
\]
The ellipse is transformed into a circle.

2. There are three equations for the second transformation. Plug in $x = \frac{u + v}{2}$ and $y = \frac{u - v}{2}$ into each one to get a function of $u$ and $v$.

(a) $y = -x + 4$

\[
y = -x + 4
\]
\[
\frac{u - v}{2} = -\frac{u + v}{2} + 4
\]
\[
u - v = -(u + v) + 8
\]
\[
u - v = -u - v + 8
\]
\[u = 4
\]

(b) $y = x + 1$

\[
y = x + 1
\]
\[
\frac{u - v}{2} = \frac{u + v}{2} + 1
\]
\[ u - v = u + v + 2 \]

\[ v = -1 \]

(c) \[ y = \frac{x}{3} - \frac{4}{3} \]

\[ y = \frac{x}{3} - \frac{4}{3} \]

\[ \frac{u - v}{2} = \frac{u + v}{6} - \frac{4}{3} \]

\[ 3(u - v) = u + v - 8 \]

\[ 3u - 3v = u + v - 8 \]

\[ 2u - 4v = -8 \]

\[ v = \frac{1}{2}u + 2 \]
Definition 1: Jacobian

Suppose \( x = g(u, v) \) and \( y = h(u, v) \). Then the Jacobian of the transformation is given by

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix}
\]

Theorem 1: Transformation

\[
\int \int_R f(x, y) \, dx \, dy = \int \int_S f(g(u, v), h(u, v)) \cdot |J| \, dA
\]

Example 2

Show that \( \int \int_R f(x, y) \, dx \, dy = \int_{\alpha}^{\beta} \int_{a}^{b} f(r \cos \theta, r \sin \theta) \cdot r \, dr \, d\theta \)

1. Let \( x = r \cos(\theta) \), and \( y = r \sin(\theta) \).
2. \( \frac{dx}{dr} = \cos(\theta) \), \( \frac{dx}{d\theta} = -r \sin(\theta) \)
3. \( \frac{dy}{dr} = \sin(\theta) \), \( \frac{dy}{d\theta} = r \cos(\theta) \)
4. Jacobian

\[
J = \begin{vmatrix}
\cos(\theta) & -r \sin(\theta) \\
\sin(\theta) & r \cos(\theta)
\end{vmatrix} = r \cos^2(\theta) + 4 \sin^2(\theta) = r
\]

Example 3

Evaluate \( \int \int_R e^{\frac{x+y}{2}} \, dA \) where \( R \) is a trapezoidal region with vertices \((1,0)\), \((2,0)\), \((0,-2)\), and \((0,-1)\). Let \( x = \frac{u+v}{2} \) and \( y = \frac{u-v}{2} \).

1. Sketch the region \( R \)
2. Equation side corresponds to a side of $S$.

(a) $y = x - 1$

\[
\frac{u - v}{2} = \frac{u + v}{2} - 1
\]

\[u - v = u + v - 2\]

\[2v = 2\]

\[v = 1\]

(b) $y = 0$

\[
\frac{u - v}{2} = 0
\]

\[u - v = 0\]

\[u = v\]

(c) $y = x - 2$

\[
\frac{u - v}{2} = \frac{u + v}{2} - 2
\]

\[u - v = u + v - 4\]

\[v = 2\]
(d) $x = 0$

\[
\frac{u + v}{2} = 0 \\
u + v = 0 \\
u = -v
\]

3. Sketch the equations from (b) to get the image $S$

4. Rewrite $e^{\frac{x+y}{\sqrt{2}}}$ as $e^u$

5. Jacobian

\[
J = \begin{vmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\]

6. Set up integral

\[
\int_1^2 \int_{-v}^v e^u \cdot J \, du \, dv
\]

(a) Evaluate the inside integral

\[
\int_{-v}^v e^{u/v} \cdot \frac{1}{2} \, du \\
= \left. \frac{v}{2} e^{u/v} \right|_{-v}^v \\
= \frac{v}{2} e - \frac{v}{2} e^{-1}
\]
(b) Evaluate the outside integral

\[
\int_1^2 \frac{v}{2} e - \frac{v}{2} e^{-1} \, dv
\]

\[
= \left. \frac{v^2}{4} e - \frac{v}{4} e^{-1} \right|_1^2
\]

\[
e - e^{-1} - \frac{1}{4} e + \frac{1}{4} e^{-1}
\]

\[
\frac{3}{4} e - \frac{3}{4} e^{-1}
\]