Math 232

Calculus III

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15.9 Change in Variables in Multiple Integrals

In one variable calculus a u substitution would have looked like this:

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$, $dx = g'(u)\,du$, $a = g(c)$, and $b = g(d)$.

Now we consider a change of variables that is given by a transformation $(x, y) = T(u, v)$ where $x = g(u, v)$ and $y = h(u, v)$.

Suppose you have a bounded region in (x, y) called R. A change in variables from (x, y) to $T(u, v)$ maps the region R to a new region S that's graphed on the uv plane. In one variable calculus this would be equivalent to interval bounds.

Example 1

Determine the new region that we get by applying the transformation to R.

1. R is the ellipse $x^2 + \dfrac{y^2}{36} = 1$ with $x = \dfrac{u}{2}$ and $y = 3v$.
2. R is the region bounded by $y = -x+4$, $y = x+1$, and $y = \dfrac{x}{3} - \dfrac{4}{3}$ with $x = \dfrac{u+v}{2}$ and $y = \dfrac{u-v}{2}$.

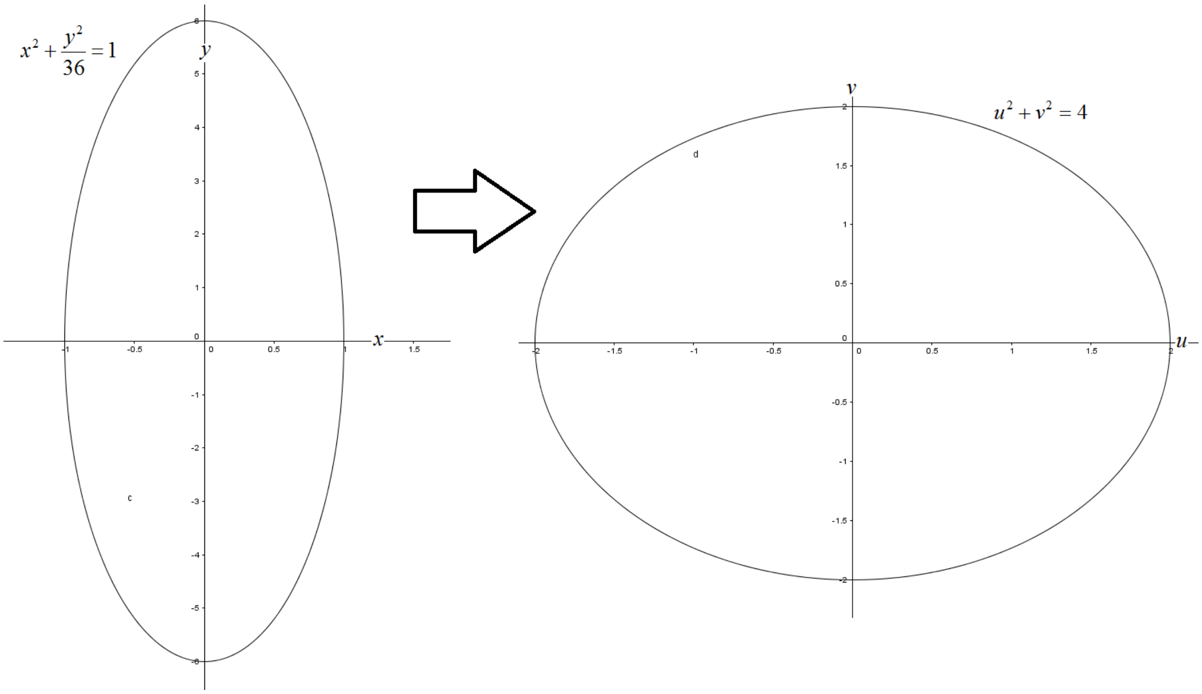
1. Take one of the equations that involve x, y, or both. In this case there is only one function $x^2 + \dfrac{y^2}{36} = 1$. Plug in $x = \dfrac{u}{2}$ and $y = 3v$.

$$x^2 + \frac{y^2}{36} = 1$$
$$\left(\frac{u}{2}\right)^2 + \frac{(3v)^2}{36} = 1$$

$$\frac{u^2}{4} + \frac{9v^2}{36} = 1$$

$$u^2 + v^2 = 4$$

The ellipse is transformed into a circle.



2. There are three equations for the second transformation. Plug in $x = \dfrac{u+v}{2}$ and $y = \dfrac{u-v}{2}$ into each one to get a function of u and v.

   (a) $y = -x + 4$

$$y = -x + 4$$

$$\frac{u-v}{2} = -\frac{u+v}{2} + 4$$

$$u - v = -(u+v) + 8$$

$$u - v = -u - v + 8$$

$$u = 4$$

   (b) $y = x + 1$

$$y = x + 1$$

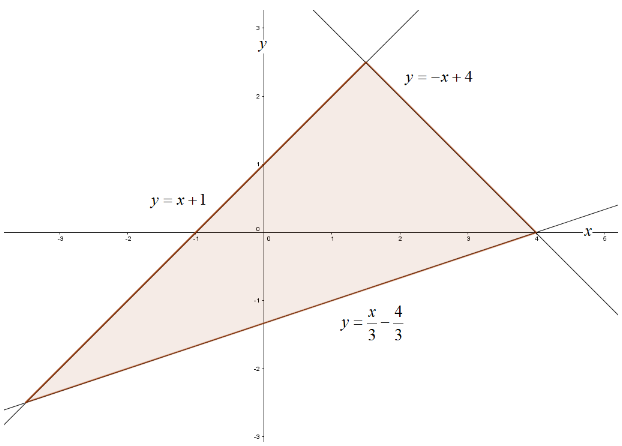
$$\frac{u-v}{2} = \frac{u+v}{2} + 1$$

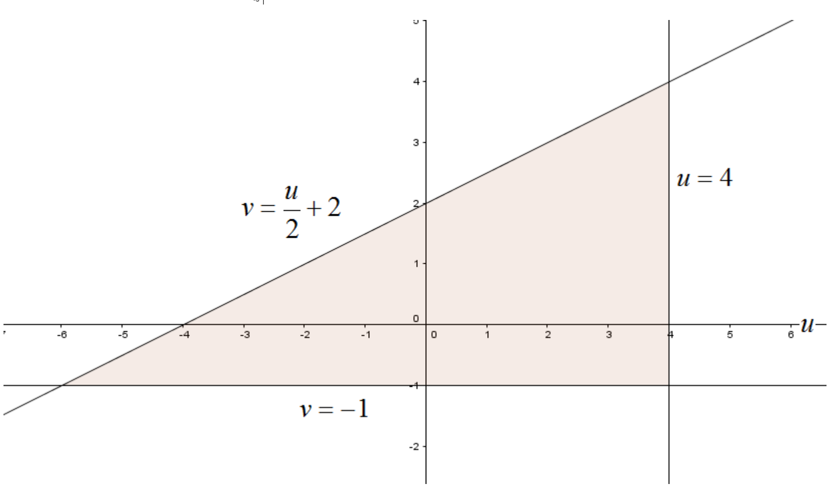
$$u - v = u + v + 2$$

$$v = -1$$

(c) $y = \frac{x}{3} - \frac{4}{3}$

$$y = \frac{x}{3} - \frac{4}{3}$$
$$\frac{u-v}{2} = \frac{u+v}{6} - \frac{4}{3}$$
$$3(u-v) = u + v - 8$$
$$3u - 3v = u + v - 8$$
$$2u - 4v = -8$$
$$v = \frac{1}{2}u + 2$$





Definition 1: Jacobian

Suppose $x = g(u, v)$ and $y = h(u, v)$. Then the Jacobian of the transformation is given by

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Theorem 1: Transformation

$$\int \int_R f(x, y) = \int \int_S f(g(u, v), h(u, v)) \cdot |J| \; dA$$

Example 2

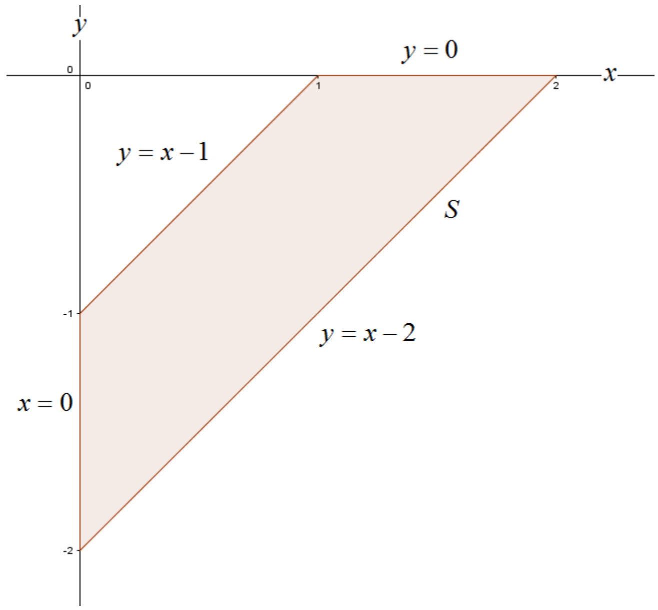
Show that $\int \int_R f(x, y) \; dx \; dy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) \cdot r \; dr \; d\theta$

1. Let $x = r\cos(\theta)$, and $y = r\sin(\theta)$.
2. $\frac{dx}{dr} = \cos(\theta)$, $\frac{dx}{d\theta} = -r\sin(\theta)$
3. $\frac{dy}{dr} = \sin(\theta)$, $\frac{dy}{d\theta} = r\cos(\theta)$
4. Jacobian
$$J = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + 4\sin^2(\theta) = r$$

Example 3

Evaluate $\int \int_R e^{\frac{x+y}{x-y}} \; dA$ where R is a trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$. Let $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$.

1. Sketch the region R



2. Equation side corresponds to a side of S.

(a) $y = x - 1$

$$\frac{u-v}{2} = \frac{u+v}{2} - 1$$
$$u - v = u + v - 2$$
$$2v = 2$$
$$v = 1$$

(b) $y = 0$

$$\frac{u-v}{2} = 0$$
$$u - v = 0$$
$$u = v$$

(c) $y = x - 2$

$$\frac{u-v}{2} = \frac{u+v}{2} - 2$$
$$u - v = u + v - 4$$
$$v = 2$$

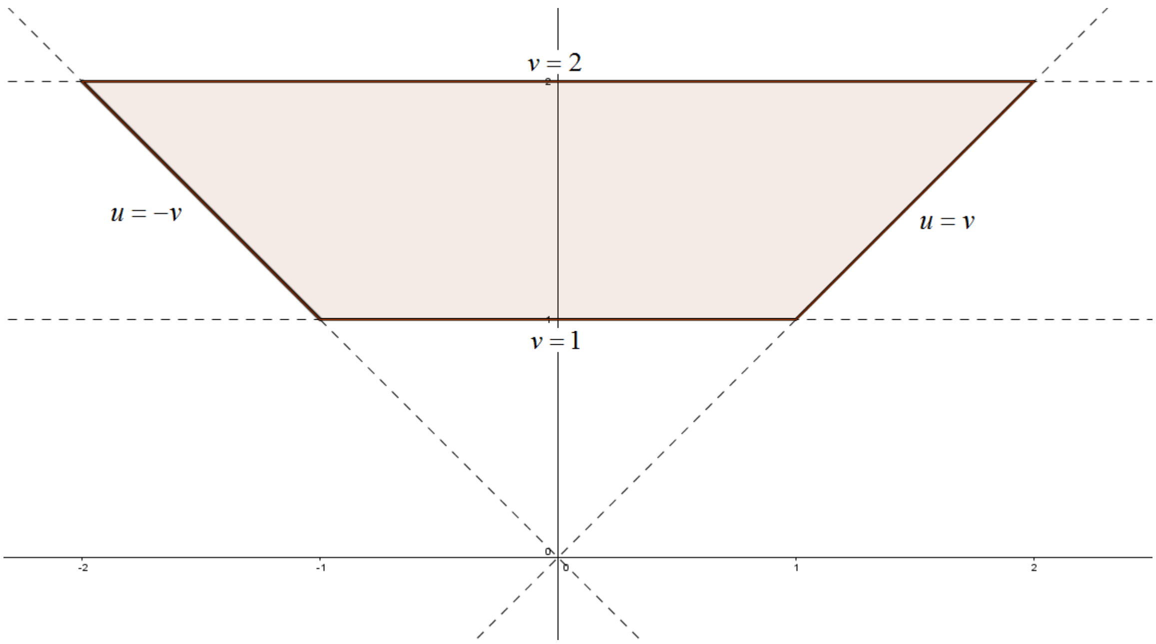
(d) $x = 0$

$$\frac{u+v}{2} = 0$$

$$u + v = 0$$

$$u = -v$$

3. Sketch the equations from (b) to get the image S



4. Rewrite $e^{\frac{x+y}{x-y}}$ as $e^{\frac{u}{v}}$

5. Jacobian

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

6. Set up integral

$$\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \cdot J \, du \, dv$$

(a) Evaluate the inside integral

$$\int_{-v}^{v} e^{u/v} \cdot \frac{1}{2} \, du$$

$$= \frac{v}{2} e^{u/v} \Big|_{-v}^{v}$$

$$= \frac{v}{2} e - \frac{v}{2} e^{-1}$$

(b) Evaluate the outside integral

$$\int_1^2 \frac{v}{2}e - \frac{v}{2}e^{-1}\, dv$$

$$= \frac{v^2}{4}e - \frac{v^2}{4}e^{-1}\Big|_1^2$$

$$e - e^{-1} - \frac{1}{4}e + \frac{1}{4}e^{-1}$$

$$\frac{3}{4}e - \frac{3}{4}e^{-1}$$