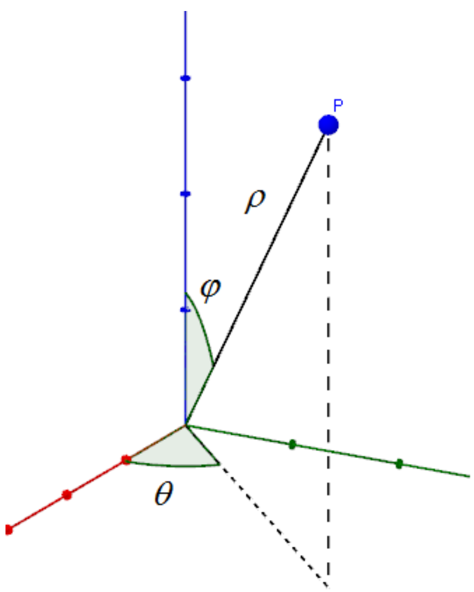
Math 232

Calculus III

Brian Veitch • Fall 2015 • Northern Illinois University

15.7 Triple Integrals in Spherical Coordinates

Definition 1: Spherical Coordinates



Convert to Cylindrical Coordinates

$$x = \rho\cos(\theta)\sin(\phi)$$

$$y = \rho\sin(\theta)\sin(\phi)$$

$$z = \rho\cos(\phi)$$

Convert to Spherical Coordinates

$$x^2 + y^2 + z^2 = \rho^2$$

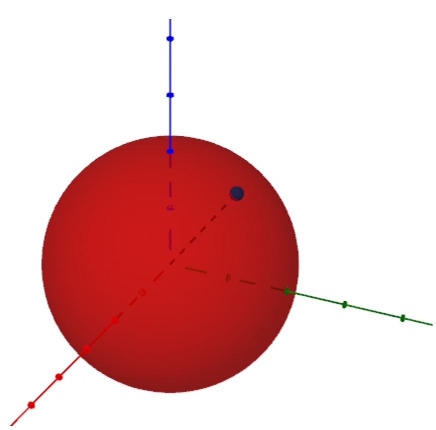
$$\cos(\phi) = \frac{z}{\rho}$$

$$\cos(\theta) = \frac{x}{\rho\sin(\phi)}$$

Example 1

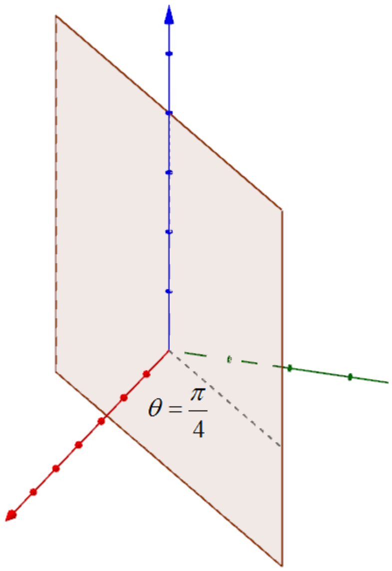
Sketch $\rho = 2$, $\theta = \pi/4$, $\phi = \pi/6$

1. $\rho = 2$



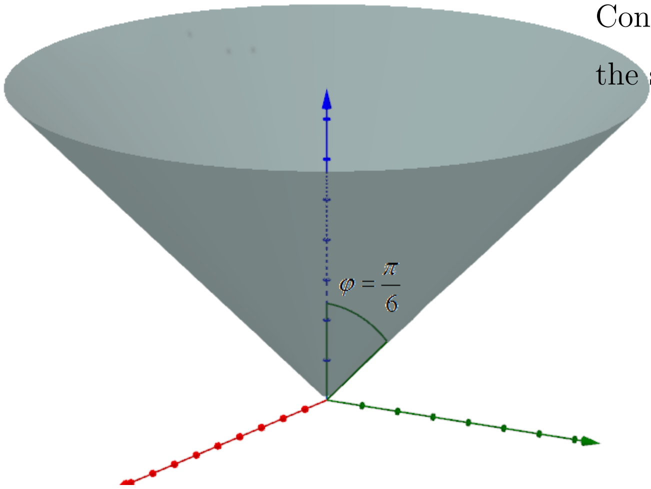
Sphere centered at $(0, 0, 0)$ with radius $\rho = 2$.

2. $\theta = \pi/4$



Plane. If you project the plane onto the xy-plane (the dotted line) the angle between the x-axis and the dotted line is $\pi/4$.

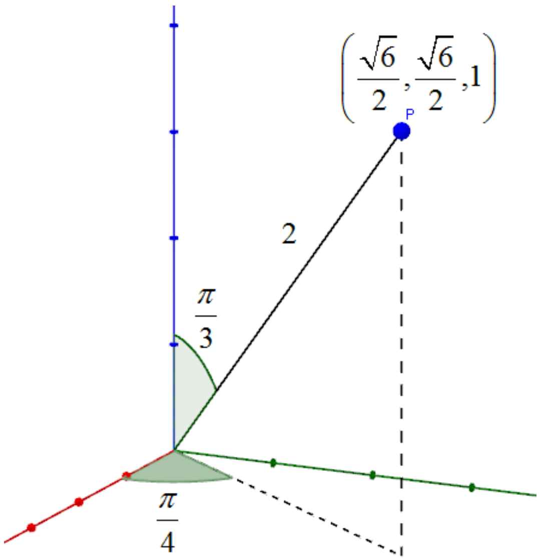
3. $\phi = \pi/6$



Cone with the angle between the z-axis to the side of the cone is $\pi/6$.

Example 2

Convert the spherical coordinates $P(2, \pi/4, \pi/3)$ to rectangular coordinates.



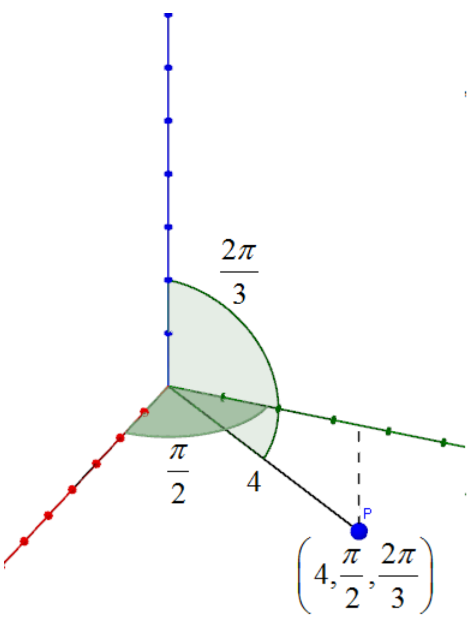
$$x = 2\cos(\pi/4)\sin(\pi/3) = 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$$

$$y = 2\sin(\pi/4)\sin(\pi/3) = 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$$

$$z = 2\cos(\pi/3) = 2\cdot\frac{1}{2} = 1$$

Example 3

Convert the rectangular coordinates $P(0, 2\sqrt{3}, -2)$ to spherical coordinates.



$$\rho = \sqrt{0^2 + (2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4$$

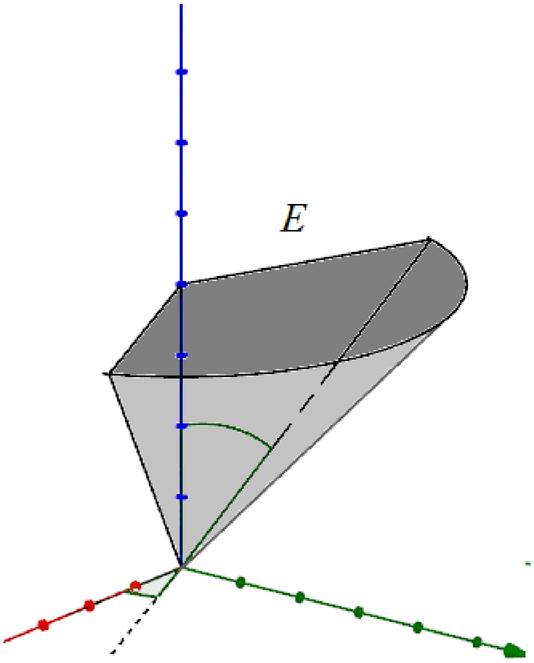
$$\cos\phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2}$$

$$\phi = 2\pi/3$$

$$\cos\theta = \frac{x}{\rho\sin\phi} = \frac{0}{4\sin(2\pi/3)} = 0$$

$$\theta = \pi/2$$

Definition 2: Spherical Change of Coordinates



$$E = \begin{cases} a \leq \rho \leq b \\ c \leq \theta \leq d \\ \gamma \leq \phi \leq \delta \end{cases}$$

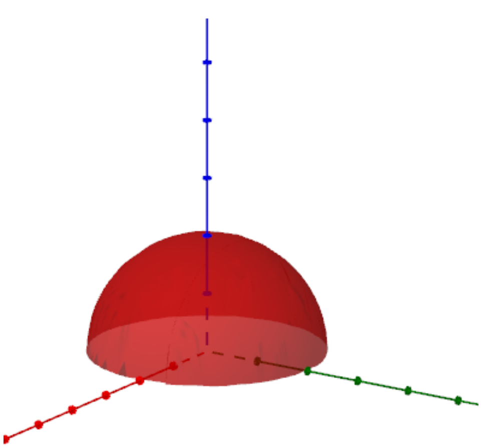
$$\int\int\int_E f(x,y,z)\, dV$$

$$= \int_a^b \int_c^d \int_\gamma^\delta f(\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \cdot \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho$$

Example 4

Evaluate $\int\int\int_E e^{(x^2+y^2+z^2)^{3/2}}\, dV$ where E is below $x^2+y^2+z^2 \leq 1$ (often referred to as top half of the unit ball) and above $z = 0$.

1. A sketch will be helpful here.



$$E = \begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/2 \end{cases}$$

2. Rewrite the function into spherical coordinates

$$e^{(x^2+y^2+y^2)^{3/2}} = e^((\rho^2)^{3/2}) = e^{\rho^3}$$

3. Set up the integral

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi/2} e^{\rho^3} \cdot \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho$$

4. Note that the integrand is a product of functions of ϕ, ρ, and θ. This means we can rewrite the integral as

$$\int_0^1 \rho^2 e^{\rho^3}\, d\rho \cdot \int_0^{2\pi} 1\, d\theta \cdot \int_0^{\pi/2} \sin(\phi)\, d\phi$$

(a) $\int_0^1 \rho^2 e^{\rho^3}\, d\rho$

i. Let $u = \rho^3$

ii. $du = 3\rho^2\, d\rho \Rightarrow \frac{1}{3}\, du = \rho^2\, d\rho$

iii. If $\rho = 0$, $u = (0)^3 = 0$

iv. If $\rho = 1$, $u = (1)^3 = 1$

v. Substitute

$$\int_0^1 \frac{1}{3} e^u\, du = \frac{1}{3} e^u \Big|_0^1 = \frac{1}{3}e - \frac{1}{3}$$

(b) $\int_0^{2\pi} 1\, d\theta$

$$\int_0^{2\pi} 1\, d\theta = \theta|_0^{2\pi} = 2\pi$$

(c) $\int_0^{\pi/2} \sin(\phi)\, d\phi$

$$\int_0^{\pi/2} \sin(\phi)\, d\phi = -\cos(\phi)|_0^{\pi/2} = 0 + 1 = 1$$

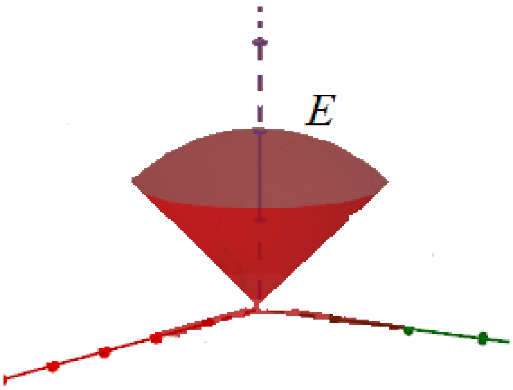
(d) Finally, multiply the three integral values together

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi/2} e^{\rho^3} \cdot \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho = \left(\frac{1}{3}e - \frac{1}{3}\right) \cdot 2\pi \cdot 1 = \frac{2}{3}\pi e - \frac{2}{3}\pi$$

Example 5

Evaluate $\iiint_E y^2z^2\,dV$, where E lies above the cone $\phi = \pi/3$ and below the sphere $x^2+y^2+z^2=1$.

1. Sketch and describe E



$$E = \begin{cases} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/3 \end{cases}$$

2. Rewrite the function so that it's in spherical coordinates

$$y^2z^2 = (\rho\sin(\theta)\sin(\phi))^2 \cdot (\rho\cos(\phi))^2 = \rho^4\sin^2(\theta)\sin^2(\phi)\cos^2(\phi)$$

3. Set up the integral

$$\int_0^1\int_0^{2\pi}\int_0^{\pi/3} \rho^4\sin^2(\theta)\sin^2(\phi)\cos^2(\phi)\cdot\rho^2\sin(\phi)\,d\phi\,d\theta\,d\rho$$

$$= \int_0^1\int_0^{2\pi}\int_0^{\pi/3} \rho^6\sin^2(\theta)\sin^3(\phi)\cos^2(\phi)\,d\phi\,d\theta\,d\rho$$

$$= \int_0^1 \rho^6\,d\rho \cdot \int_0^{2\pi}\sin^2(\theta)\,d\theta \cdot \int_0^{\pi/3}\sin^3(\phi)\cos^2(\phi)\,d\phi$$

(a) $\int_0^1 \rho^6\,d\rho$

$$\int_0^1 \rho^6\,d\rho = \frac{1}{7}rho^7\Big|_0^1 = \frac{1}{7}$$

(b) $\int_0^{2\pi} \sin^2(\theta)\ d\theta$

$$\int_0^{2\pi} \sin^2(\theta)\ d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(2\theta)\ d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\Big|_0^{2\pi} = \pi$$

(c) $\int_0^{\pi/3} \sin^3(\phi)\cos^2(\phi)\ d\phi$

$$\int_0^{\pi/3} \sin^3(\phi)\cos^2(\phi)\ d\phi = \int_0^{\pi/3} \sin^2(\phi)\cos^2(\phi)\cdot\sin(\phi)\ d\phi$$

$$= \int_0^{\pi/3} (1-\cos^2(\phi))\cos^2(\phi)\cdot\sin(\phi)\ d\phi$$

i. Let $u = \cos(\phi)$

ii. $du = -\sin(\phi)\ d\phi \Rightarrow -\ du = \sin(\phi)\ d\phi$

iii. If $\phi = \pi/3$, $u = \cos(\pi/3) = 1/2$

iv. If $\phi = 0$, $u = \cos(0) = 1$

v. Substitute

$$\int_1^{1/2} -(1-u^2)u^2\ du = \int_1^{1/2} u^4 - u^2\ du = \frac{1}{5}u^5 - \frac{1}{3}u^3\Big|_0^{\pi/3} = \frac{-17}{480} + \frac{2}{15} = \frac{47}{480}$$

4. Final Answer:

$$= \int_0^1 \rho^6\ d\rho \cdot \int_0^{2\pi} \sin^2(\theta)\ d\theta \cdot \int_0^{\pi/3} \sin^3(\phi)\cos^2(\phi)\ d\phi = \frac{1}{7}\cdot\pi\cdot\frac{47}{480} = \frac{47}{3360}\pi$$

Definition 3: Finding the Volume of E

Find the volume of E. Let $f(x,y,z) = 1$.

$$V = \int_a^b \int_c^d \int_\gamma^\delta 1\cdot\rho^2\sin(\phi)\ d\phi\ d\theta\ d\rho$$