Math 232

Calculus III

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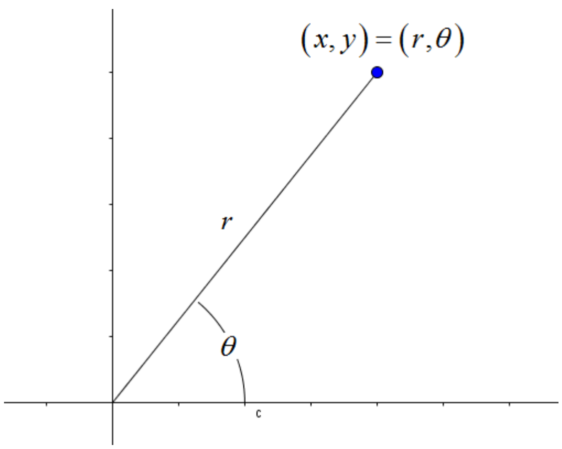
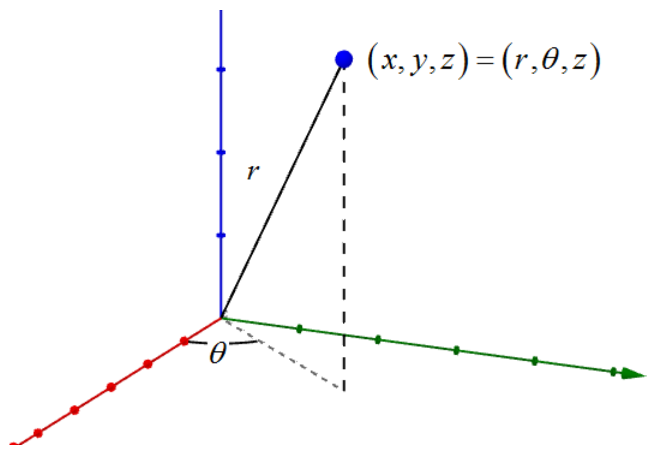
15.7 Triple Integrals in Cylindrical Coordinates

In 2D this would be called Polar Coordinates. When extending it to 3D, by adding the z-axis, we represent points (x, y, z) as (r, θ, z).





Definition 1: Convert Coordinates

Cylindrical to Rectangular Coordinates

$$x = r\cos(\theta),\ y = r\sin(\theta),\ z = z$$

Rectangular to Cylindrical Coordinates

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x},\ z = z$$

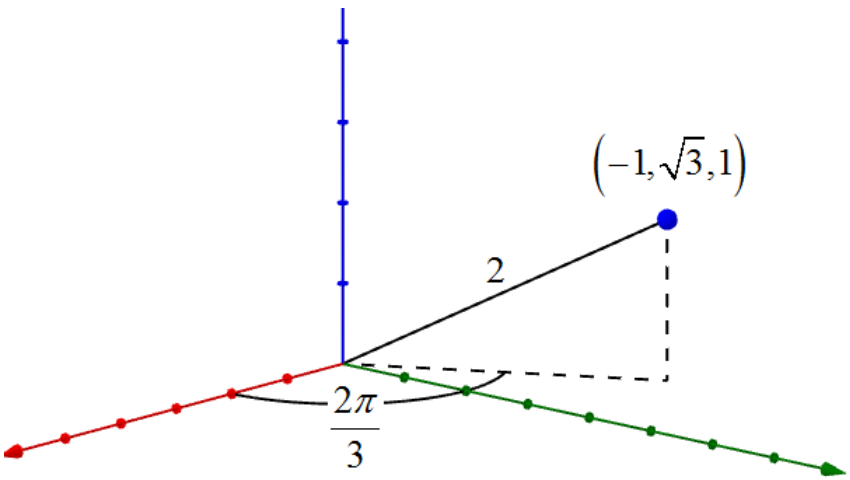
Example 1

Plot $\left(2, \frac{2\pi}{3}, 1\right)$ and find the rectangular coordinates.

$$x = r\cos(\theta) = 2\cos(2\pi/3) = 2(-1/2) = -1$$

$$y = r\sin(\theta) = 2\sin(2\pi/3) = 2(\sqrt{3}2) = \sqrt{3}$$

$$z = 1$$



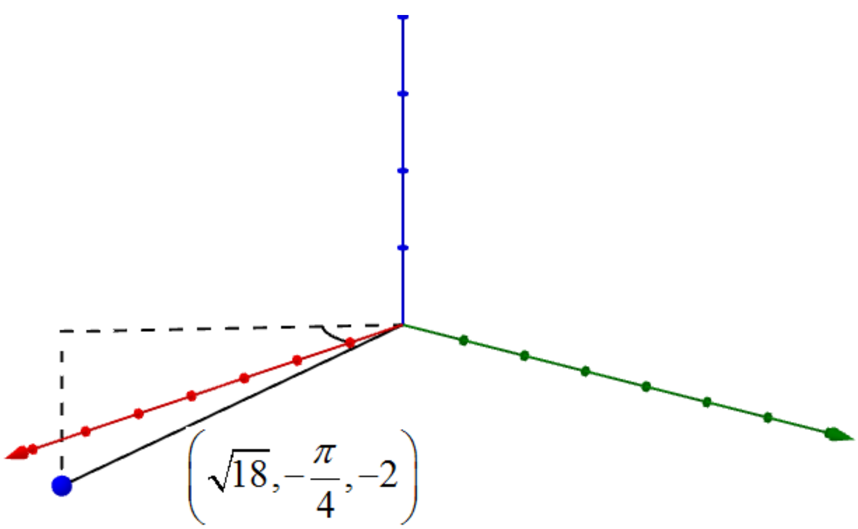
Example 2

Change to Cylindrical Coordinates: $(3, -3, -2)$

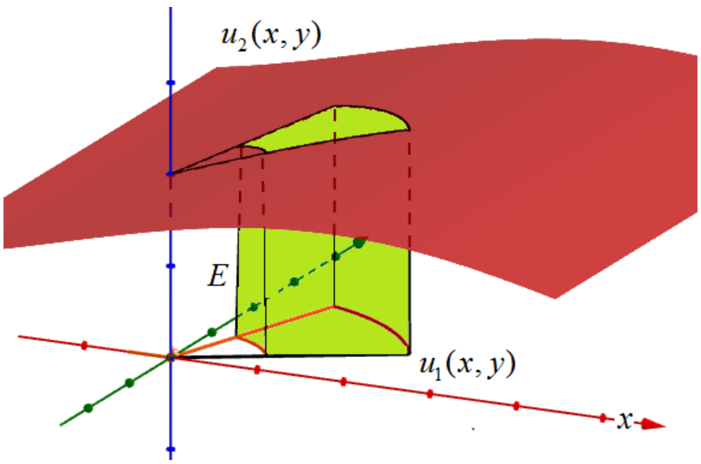
$$r^2 = (3)^2 + (-3)^2 = 18 \Rightarrow r = \sqrt{18}$$

$$\tan(\theta) = \frac{3}{-3} = -1 \Rightarrow \theta = -\pi/4$$

$$z = -2$$



Definition 2: Triple Integrals in Cylindrical Coordinates



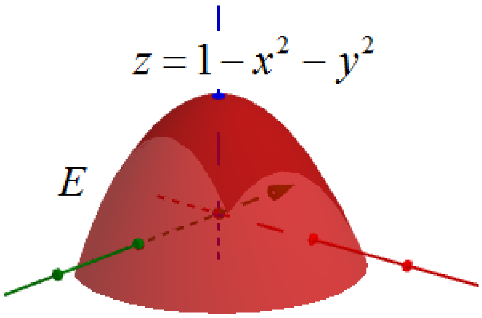
$$E = \begin{cases} \alpha \leq \theta \leq \beta \\ r_1(\theta) \leq r \leq r_2(\theta) \\ u_1(r\cos\theta, r\sin\theta) \leq z \leq u_2(r\cos\theta, r\sin\theta) \end{cases}$$

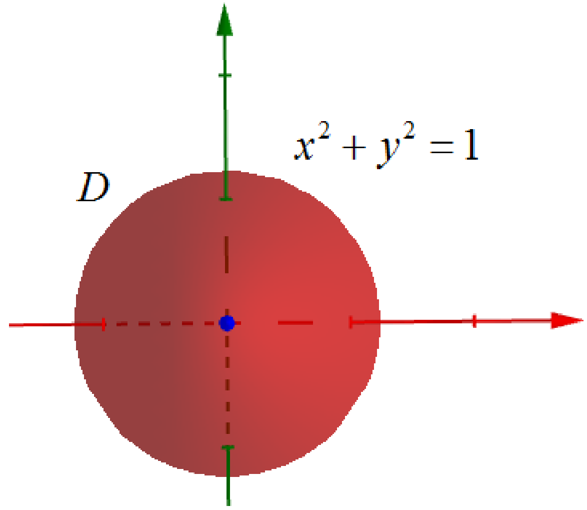
$$\int\int\int_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta) r\, dz\, dr\, d\theta$$

Example 3

Find the volume of the potion of the surface that lies below $z = 1 - x^2 - y^2$ and above the xy-plane.

Let's take a look at the graph and the projection of E onto the xy plane.





We are trying to evaluate $\int\int\int_E 1\ dV$

1. Let's start with trying to write E

$$E = \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \\ 0 \leq z \leq 1 - x^2 - y^2 \end{cases}$$

Notice how the inequalities involve x and y. If we want to convert this triple integral to cylindrical coordinates we need to rewrite x and y using the conversion formulas from above.

$$1 - x^2 - y^2 = 1 - (x^2 + y^2) = 1 - r^2$$

2. Now express E as

$$E = \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \\ 0 \leq z \leq 1 - r^2 \end{cases}$$

3. Set up the integral. Recall that if you're looking for th volume then $f(x, y, z) = 1$

$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} 1r\ dz\ dr\ d\theta$$

4. Evaluate the inside integral

$$\int_0^{1-r^2} 1r\,dz = 1rz\Big|_0^{1-r^2} = r(1-r^2) = r - r^3$$

$$\int_0^{2\pi}\int_0^1 r - r^3\,dr\,d\theta$$

5. Evaluate the inside integral

$$\int_0^1 r - r^3\,dr = \frac{1}{2}r^2 - \frac{1}{4}r^4\Big|_0^1 = \frac{1}{4}$$

$$\int_0^{2\pi} \frac{1}{4}\,d\theta$$

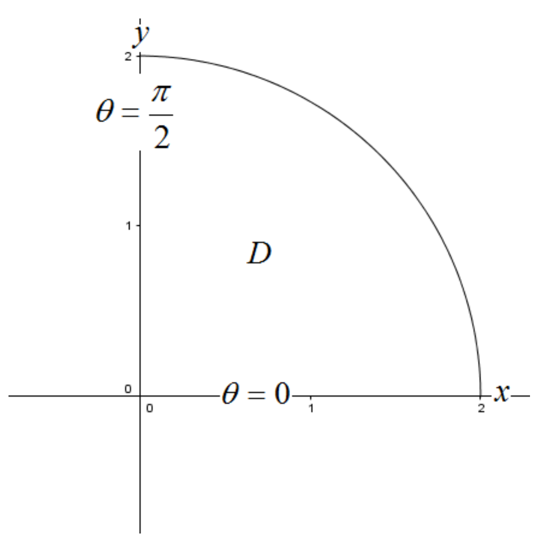
6. Evaluate the last integral

$$\int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{1}{4}\theta\Big|_0^{2\pi} = \frac{\pi}{2}$$

Example 4

Setup $\int\int\int_E x + y + z\,dV$ where E is the solid in the first octant that lies under $z = 4 - x^2 - y^2$ as a triple integral in cylindrical coordinates.

If you want to project the surface onto the xy plane, you get



$$D = \begin{cases} 0 \le \theta \le \pi/2 \\ 0 \le r \le 2 \end{cases}$$

1. Now we can express E as

$$E = \begin{cases} 0 \le \theta \le \pi/2 \\ 0 \le r \le 2 \\ 0 \le z \le 4 - x^2 - y^2 \end{cases}$$

2. Covert to Cylindrical Coordinates

$$E = \begin{cases} 0 \le \theta \le \pi/2 \\ 0 \le r \le 2 \\ 0 \le z \le 4 - r^2 \end{cases}$$

3. The integral can be written as

$$\int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\, r \, dz\, dr\, d\theta$$

$$\int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} r^2\cos\theta + r^2\sin\theta + zr \, dz\, dr\, d\theta$$

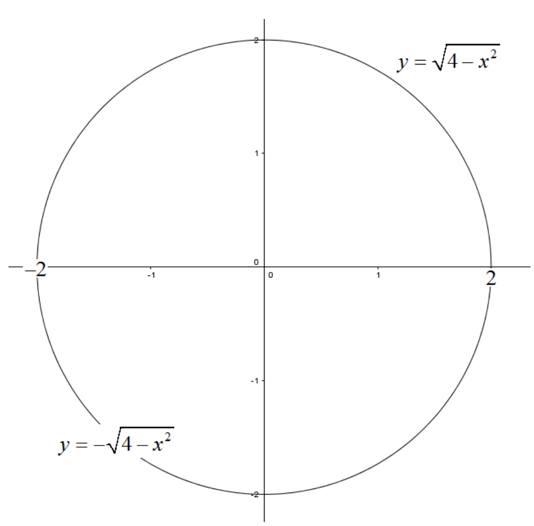
Example 5

Sketch the region E represented by the integral

$$\int\int\int_E x^2 + y^2 \, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} x^2 + y^2 \, dz\, dy\, dx$$

Then Evaluate the integral in cylindrical coordinates.

The projection of E onto the xy plane will be



$$D = \begin{cases} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{cases}$$

We can rewrite the projection region D in cylindrical coordinates by

$$D = \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{cases}$$

Adding in z the region E is expressed as

$$E = \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \\ \sqrt{x^2 + y^2} \le z \le 2 \end{cases}$$

$$E = \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \\ r \le z \le 2 \end{cases}$$

$$\int_0^{2\pi} \int_0^2 \int_r^2 (r^2) r \; dz \; dr \; d\theta$$
$$= \int_0^{2\pi} \int_0^2 \int_r^2 r^3 \; dz \; dr \; d\theta$$

1. Evaluate inside integral

$$\int_r^2 r^3 \; dz = r^3 z\Big|_r^2 = 2r^3 - r^4$$

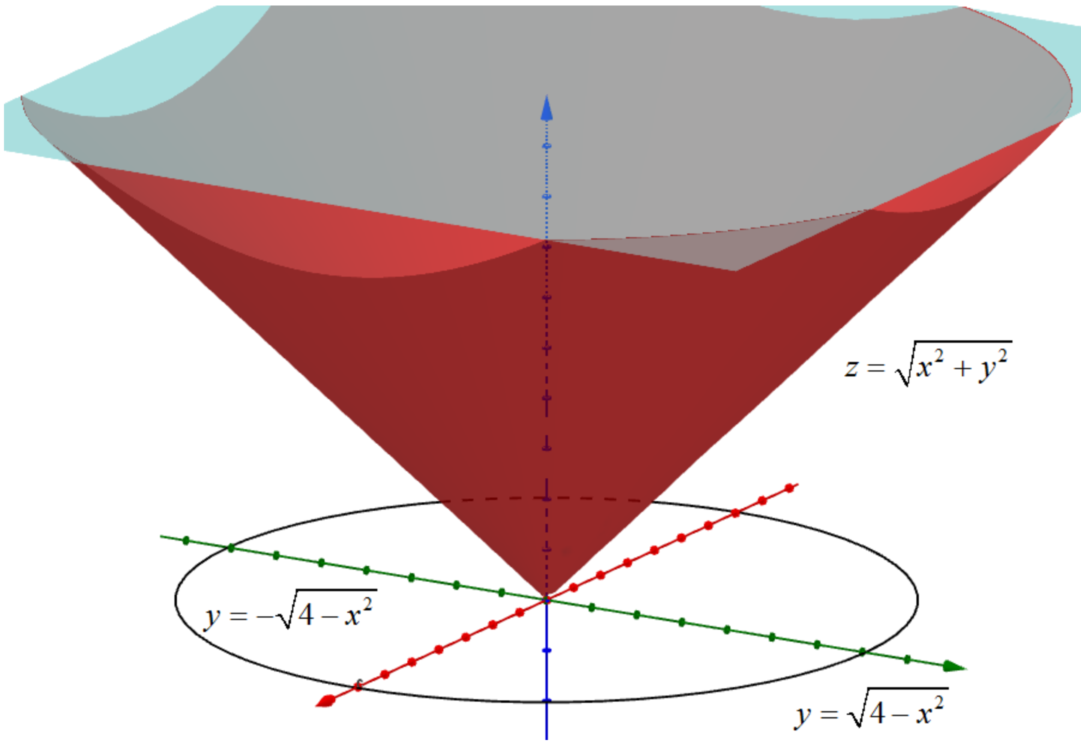
$$\int_0^{2\pi} \int_0^2 2r^3 - r^4 \; dr$$

2. Evaluate the inside integral

$$\int_0^2 2r^3 - r^4 \; dr = \frac{1}{2}r^4 - \frac{1}{5}r^5\Big|_0^2 = \frac{8}{5}$$

$$\int_0^{2\pi} \frac{8}{5} \; d\theta$$

$$= \frac{8}{5}\theta\Big|_0^{2\pi} = \frac{16\pi}{5}$$


$z = \sqrt{x^2 + y^2}$
$y = -\sqrt{4 - x^2}$
$y = \sqrt{4 - x^2}$