Math 232

Calculus III

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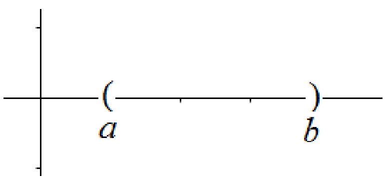
15.6 Triple Integrals

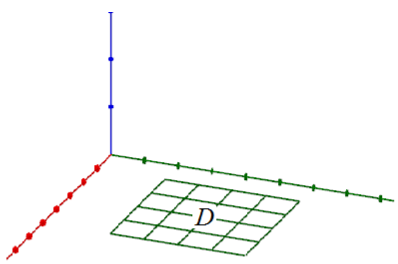
In order to build up to a triple integral let's start back at an integral in one variable. From here we'll extend the concept to a triple integral.

1. Single Integral - the domain is the integral I (a line)



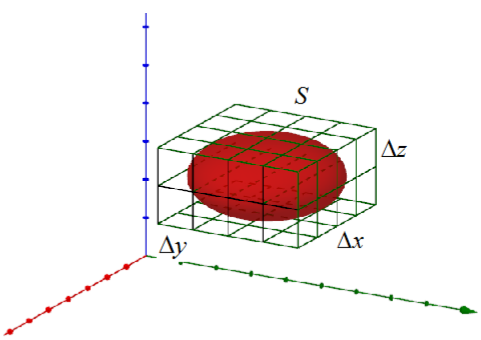
$$\sum_{\substack{\text{over}\\\text{sub}\\\text{intervals}}} f(x)\Delta x \;\longrightarrow\; \int_I f(x)dx$$

2. Double Integral - the domain is D (a 2D surface).



$$\sum\sum_{\substack{\text{over}\\\text{sub}\\\text{rectangles}}} f(x_i, y_j)\Delta x\Delta y \;\longrightarrow\; \iint_D f(x,y)dA$$

3. Triple Integral - the domain is S (a 3D solid).



$$\sum\sum\sum_{\substack{\text{over}\\\text{sub}\\\text{boxes}}} f(x_i, y_j, z_k)\Delta x\Delta y\Delta z \;\longrightarrow\; \iiint_S f(x,y,z)dV$$

All the properties from single variable integrals and double integrals extend to triple integrals.

$\displaystyle\int_S\int\int f(x,y,z)\,dV =$ the net signed 4D hyper volume of the hyper solid formed between the xyz space and the hyper sufrace $w = f(x,y,z)$. The geometric interpretation of a triple integral is not easy to explain. We are mainly interested in

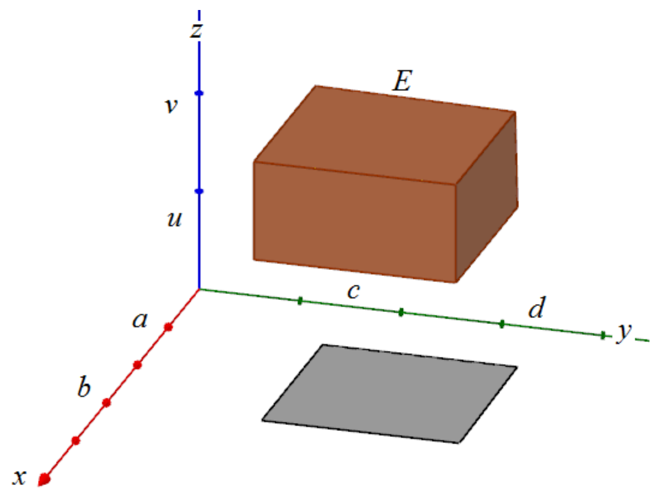
1. Are we able to extend integrals to three variables? Yes.

2. Are there applications of triple integrals? Also yes.

Definition 1: Fubini's Theorem of Rectangles

Suppose $f(x,y,z)$ is continuous on the domain

$$E = \{(x,y,z) \mid a \le b,\ c \le y \le d,\ u \le z \le v\} = [a,b] \times [c,d] \times [u,v]$$



Fubini's Theorem states that this the integral $\displaystyle\int\int_E\int f(x,y,z)\,dV$ can be written six different ways.

$$\int_a^b\int_c^d\int_u^v f\;dz\;dy\;dx \qquad \int_c^d\int_a^b\int_u^v f\;dz\;dx\;dy \qquad \int_u^v\int_a^b\int_c^d f\;dy\;dx\;dz$$

$$\int_a^b\int_u^v\int_c^d f\;dy\;dz\;dx \qquad \int_c^d\int_u^v\int_a^b f\;dx\;dz\;dy \qquad \int_u^v\int_c^d\int_a^b f\;dx\;dy\;dz$$

Example 1

Evaluate $\displaystyle\iiint_E x + yz\, dV$ where $E = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ 1 \leq z \leq 2\}$

According to Fubini's Theorem, we have six different ways the triple integral can be set up. We will choose two different ones to try.

1. $\displaystyle\int_0^1 \int_0^2 \int_1^2 x + yz\, dz\, dy\, dx$

$$\int_1^2 x + yz\, dz = xz + \frac{1}{2}yz^2\Big|_1^2 = (2x + 2y) - (x + y/2) = x + \frac{3}{2}y$$

$$\int_0^2 x + \frac{3}{2}y\, dy = xy + \frac{3}{4}y^2\Big|_0^2 = 2x + 3$$

$$\int_0^1 2x + 3\, dx = x^2 + 3x\big|_0^1 = 4$$

2. $\displaystyle\int_1^2 \int_0^2 \int_0^1 x + yz\, dx\, dy\, dz$

$$\int_0^1 x + yz\, dx = \frac{1}{2}x^2 + xyz\Big|_0^1 = (\frac{1}{2} + yz) - (0 + 0) = \frac{1}{2} + yz$$

$$\int_0^2 \frac{1}{2} + yz\, dy = \frac{1}{2}y + \frac{1}{2}y^2 z\Big|_0^2 = 1 + 2z$$

$$\int_1^2 1 + 2z\, dz = z + z^2\big|_1^2 = 6 - 2 = 4$$

Example 2

Evaluate $\displaystyle\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy\, dz\, dx\, dy$

1. Start with the inner integral

$$\int_0^{x+y} 6xy\, dz = 6xyz\big|_0^{x+y} = 6xy(x + y) = 6x^2y + 6xy^2$$

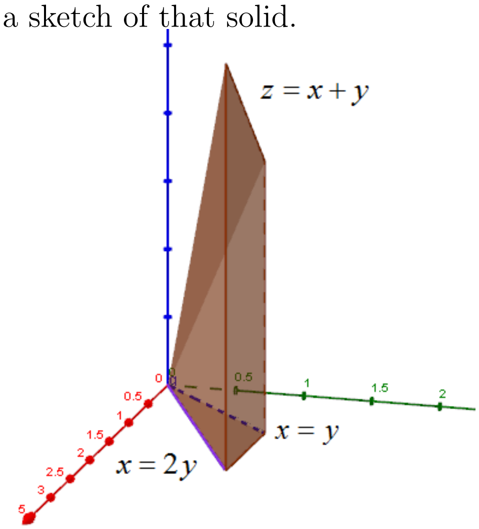
2. Now we have $\int_0^1 \int_y^{2y} 6x^2y + 6xy^2 \; dx \; dy$

$$\int_y^{2y} 6x^2y + 6xy^2 \; dx = 2x^3y + 3x^2y^2\Big|_y^{2y}$$
$$= \left[2(2y)^3y + 3(2y)^2y^2\right] - \left[2(y)^3y + 3(y)^2y^2\right]$$
$$= 16y^4 + 12y^4 - 2y^4 - 3y^4 = 23y^4$$

3. Now we have $\int_0^1 23y^4 \; dy$

$$\int_0^1 23y^4 \; dy = \frac{23}{5}y^5\Big|_0^1 = \frac{23}{5}$$

The previous example was an example of integrating over a general region (solid). Below is a sketch of that solid.

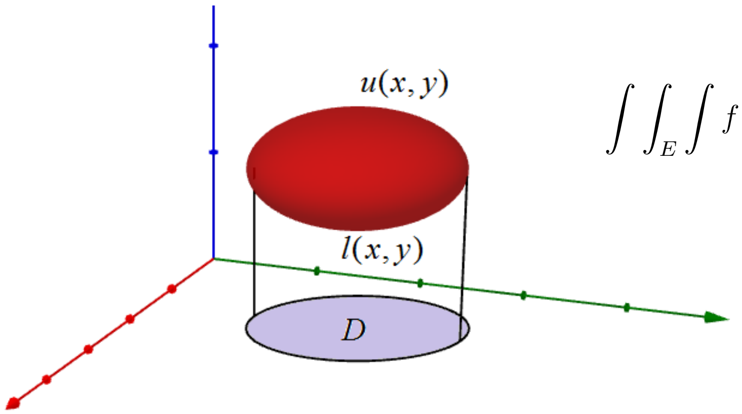


Now the question is... If I give you the solid, can you describe it as a triple integral?

Over a General Bounded Region

Definition 2: z Simple - Type 1 Solid

A solid region E is said to be z Simple (Type 1) if z lies between two functions of x and y, $u(x, y)$ and $l(x, y)$ where D is the projection of E onto the xy plane.



$$\int\int_E\int f\,dV = \int_D\int\left[\int_{l(x,y)}^{u(x,y)} f\,dz\right]dA$$

Now that you have a visual on D, write D like you did in the previous section about general regions. Doing so will give you two possible integrals

$$\int_a^b\int_{b(x)}^{t(x)}\int_{l(x,y)}^{u(x,y)} f(x, y, z)\,dz\,dy\,dx$$

$$E = \{(x, y, z) \mid a \le x \le b,\ b(x) \le y \le t(x),\ l(x, y) \le z \le u(x, y)\}$$

OR

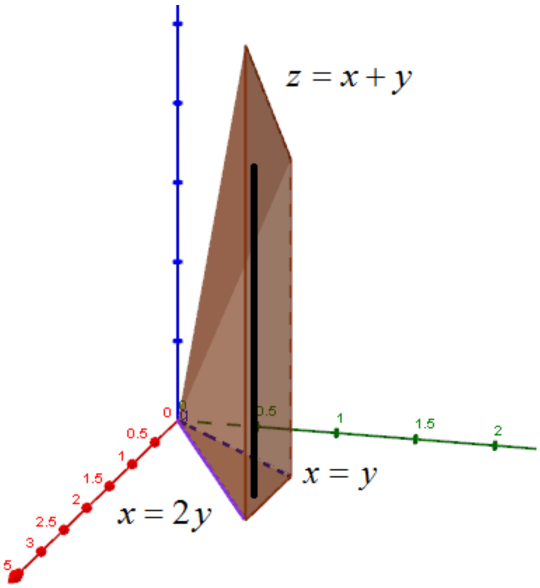
$$\int_c^d\int_{l(y)}^{r(y)}\int_{l(x,y)}^{u(x,y)} f(x, y, z)\,dz\,dx\,dy$$

$$E = \{(x, y, z) \mid c \le y \le d,\ l(y) \le x \le r(x),\ l(x, y) \le z \le u(x, y)\}$$

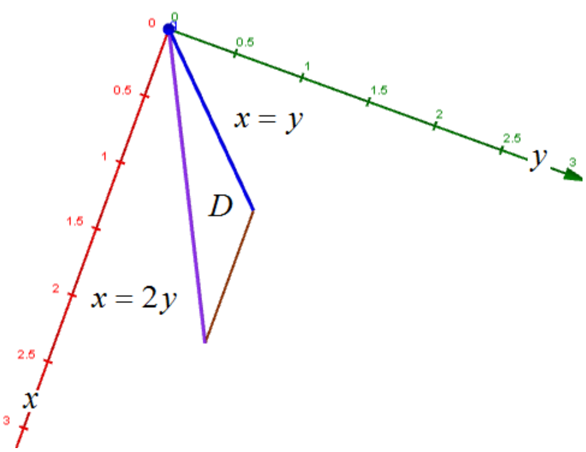
Example 3

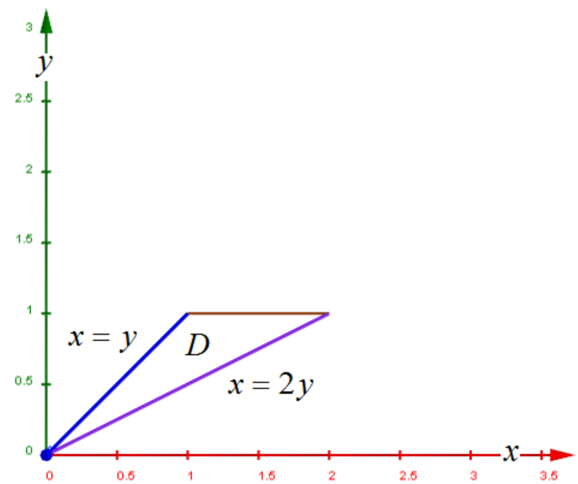
Evaluate $\int\int_E\int 6xy\ dV$ where E is bounded below by $z = x + y$, the xy plane, $x = 2y$, $x = y$ and $y = 1$.

The hardest thing is trying to get the sketch. The region E actually is



Notice that it is vertically simple (z is always between the xy plane $z = 0$ and the plane $z = x + y$). Now we project the region onto the xy plane. It looks like





The right graph is the projection of E onto the xy plane with the axes rotated to our standard position.

Since D is horizontally simple, we describe D as

$$D = \{(x, y) \mid 0 \leq y \leq 1,\ y \leq x \leq 2y\}$$

This means we can describe E as

$$E = \{(x, y, z) \mid 0 \le y \le 1,\ y \le x \le 2y,\ 0 \le z \le x + y\}$$

Following the setup from above,

$$\int\int\int_E 6xy\ dV = \int_0^1 \int_y^{2y} \int_0^{x+y} 6xy\ dz\ dx\ dy$$

Example 4

Setup the triple integral for $\int\int\int_E e^{z/y}\ dV$ where

$$E = \left\{(x, y, z) \mid 1 \le z \le 4,\ \sqrt{z} \le y \le z^2,\ z + y \le x \le 2z + y\right\}$$

$$\int_1^4 \int_{\sqrt{z}}^{z^2} \int_{z+y}^{2z+y} e^{z/y}\ dx\ dy\ dz$$

Definition 3: x Simple - Type 2 Solid

Let $E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \le x \le u_2(y, z)\}$

where D is the projection of E onto the yz plane.

$$\int\int\int_E f(x, y, z)\ dV = \int_D\int \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\ dx\right] dA$$

Definition 4: y Simple - Type 3 Solid

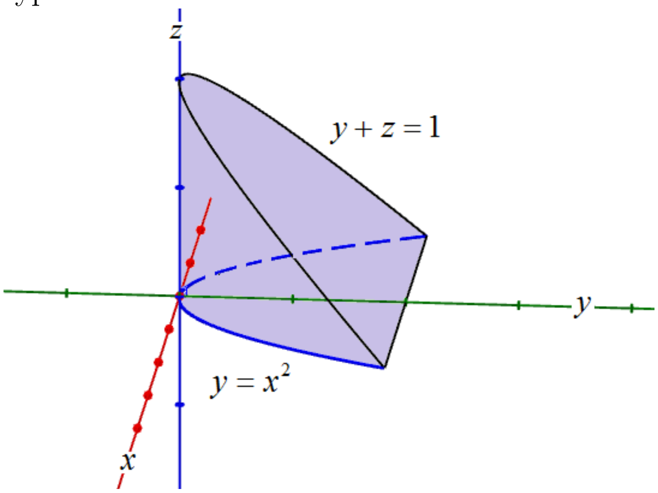
Let $E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\}$

where D is the projection of E onto the xz plane.

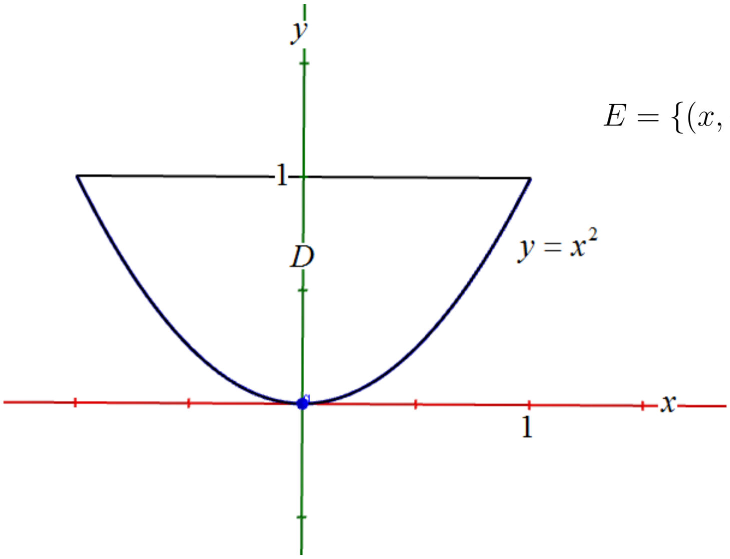
$$\int\int\int_E f(x, y, z)\ dV = \int_D\int \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\ dy\right] dA$$

Example 5

Setup the integral of $\int\int\int_E z\, dV$ where D is the region bounded by the xy plane, the parabolic cylinder $y = x^2$, and $y + z = 1$. Set it up using two of the three solid types.



1. The solid is z simple. z stays between $z = 0$ and $z = 1 - y$. This means we need to project the solid E onto the xy plane to get D. If you do that, you get the following graph



$$E = \{(x, y, z) \mid (x, y) \in D,\ 0 \le z \le 1 - z\}$$

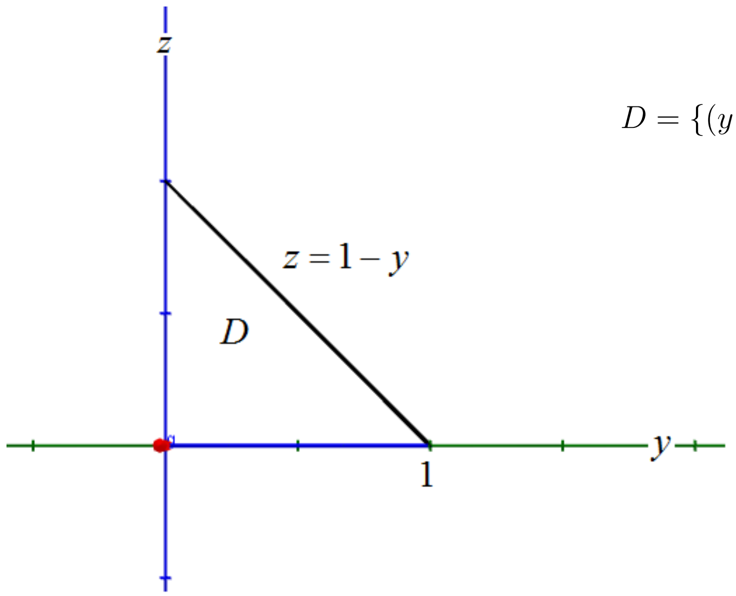
The next step is to describe the region D. It's vertically simple so we can write D as

$$D = \{(x,y) \mid -1 \leq x \leq 1,\ x^2 \leq y \leq 1\}$$

The final triple integral is

$$\int\int\int_E z\, dV = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} z\, dz\, dy\, dx$$

2. The solid is x simple. x stays between the functions $x = -\sqrt{y}$ and $x = \sqrt{y}$. This means we project E onto the yz plane. If you do that, you get the following graph



$$D = \{(y,z) \mid 0 \leq y \leq 1,\ 0 \leq z \leq 1-y\}$$

The final triple integral is

$$\int\int\int_E z\, dV = \int_{0}^{1}\int_{0}^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} z\, dx\, dz\, dy$$

Definition 5: Volume of a Solid

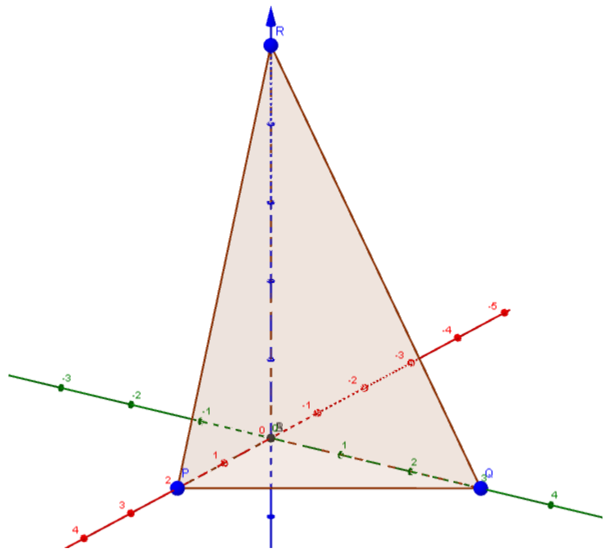
Suppose $f(x,y,z) = 1$. Then the triple integral

$$\int\int\int_E 1\, dV = V(E)$$

where $V(E)$ is the volume of the solid E.

Example 6

Find the volume fo a tetrahedron E with coordinates $(0,0,0)$, $(2,0,0)$, $(0,3,0)$, and $(0,0,5)$.



The solid is z simple, y simple and x simple. The easiest one is the z simple form. From the graph we can see that z is between $z = 0$ and $z =$ plane. Let's find the equation of that plane.

1. Assume the equation of the plane has the form $a(x-2)+b(y-0)+c(z-0) = 0$. We need to find the normal vector $\langle a, b, c\rangle$.

2. To find $\vec{n} = \langle a, b, c\rangle$, we need to find the cross product of two vectors on the plane.

$$\vec{PQ} = \langle -2, 3, 0\rangle$$

$$\vec{PR} = \langle -2, 0, 5\rangle$$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ -2 & 0 & 5 \end{vmatrix} = i\begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} - j\begin{vmatrix} -2 & 0 \\ -2 & 5 \end{vmatrix} + k\begin{vmatrix} -2 & 3 \\ -2 & 0 \end{vmatrix} = 15i + 10j + 6k$$

3. The equation of the plane is

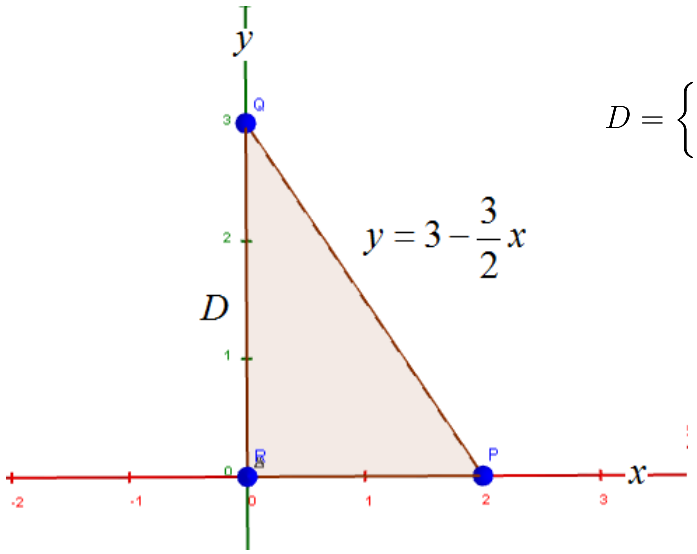
$$15(x-2) + 10y + 6z = 0$$

$$15x + 10y + 6z = 30$$
$$\Rightarrow z = 5 - \frac{5}{2}x - \frac{5}{3}y$$

4. So far we have

$$E = \left\{(x, y, z) \mid (x, y) \in D,\ 0 \le z \le 5 - \frac{5}{2}x - \frac{5}{3}y\right\}$$

5. When you project E onto the xy plane we get



$$D = \left\{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 3 - \frac{3}{2}x\right\}$$

6. Putting this all together we have the volume

$$\int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{5-\frac{5}{2}x-\frac{5}{3}y} 1\ dz\ dy\ dx$$

(a) Evaluate the inner integral

$$\int_0^{5-\frac{5}{2}x-\frac{5}{3}y} 1\ dz = z\Big|_0^{5-\frac{5}{2}x-\frac{5}{3}y}$$
$$= 5 - \frac{5}{2}x - \frac{5}{3}y$$

(b) Move to the next integral

$$\int_0^{3-\frac{3}{2}x} 5 - \frac{5}{2}x - \frac{5}{3}y\ dy = 5y - \frac{5}{2}xy - \frac{5}{6}y^2\Big|_0^{3-\frac{3}{2}x}$$

$$= 5(3 - 3x/2) - \frac{5}{2}x(3 - 3x/2) - \frac{5}{6}(3 - 3x/2)^2$$
$$= 15 - \frac{15}{2}x - \frac{15}{2}x + \frac{15}{4}x^2 - \frac{5}{6}(9 - 9x + 9x^2/4)$$
$$= \frac{15}{8}x^2 - \frac{15}{2}x + \frac{15}{2}$$

(c) Last integral

$$\int_0^2 \frac{15}{8}x^2 - \frac{15}{2}x + \frac{15}{2}\ dx$$
$$\left.\frac{15}{24}x^3 - \frac{15}{4}x^2 + \frac{15}{2}x\right|_0^2$$
$$= 5$$