Math 232

Calculus III

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15.4 Applications of Double Integrals

Suppose a lamina occures a region D of the xy plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D.

Definition 1: Mass

$$\text{mass } = m = \int\int_D \rho(x, y)\, dA$$

Moments and Centers of Mass

The moment of a particle about an axis is the product of its mass and its directed distance from the axis.

Definition 2: Moment of the lamina

About the x-axis

$$M_x = \int\int_D y\rho(x, y)\, dA$$

About the y-axis

$$M_y = \int\int_D x\rho(x, y)\, dA$$

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying region D and having density $\rho(x, y)$ are

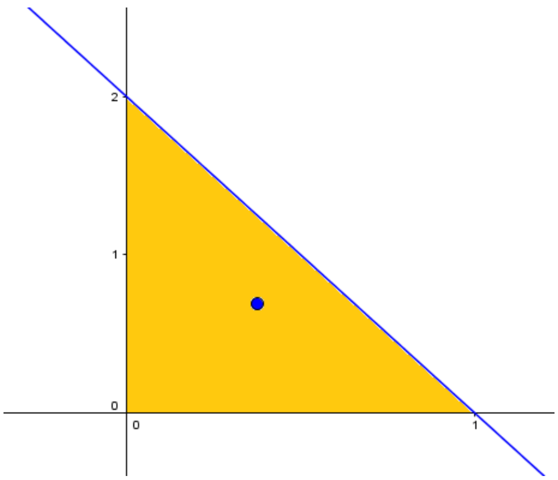
$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\int\int_D x\rho(x, y)\, dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\int\int_D y\rho(x, y)\, dA$$

Example 1

Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

1. Let's take a look at the region D and describe it



$$D = \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 2 - 2x \end{cases}$$

2. Let's find the mass m

$$\begin{aligned} m &= \int\int_D \rho(x, y)\, dA \\ &= \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx \\ &= \int_0^1 \left[y + 3xy + \frac{1}{2}y^2 \Big|_0^{2-2x} \right] dx \\ &= \int_0^1 \left[(2 - 2x) + 3x(2 - 2x) + \frac{1}{2}(2 - 2x)^2 \right] dx \\ &= \int_0^1 -4x^2 + 4\, dx \\ &= -\frac{4}{3}x^3 + 4x \Big|_0^1 \\ &= \frac{8}{3} \end{aligned}$$

3. Let's find the center of mass $(\bar{x}, \bar{y})$.

$$\begin{aligned}\bar{x} &= \frac{1}{m}\int\int_D x\rho(x,y)\,dA \\ &= \frac{3}{8}\int_0^1\int_0^{2-2x}(x+3x^2+xy)\,dy\,dx\end{aligned}$$

(a) Let's evaluate the inside integral

$$\begin{aligned}\int_0^{2-2x} x+3x^2+xy\,dy &= xy+3x^2y+\frac{1}{2}xy^2\Big|_0^{2-2x} \\ &= x(2-2x)+3x^2(2-2x)+\frac{1}{2}x(2-2x)^2 \\ &= 2x-2x^2+6x^2-6x^3+\frac{1}{2}x(4-8x+4x^2) \\ &= 4x-4x^3\end{aligned}$$

(b) Evaluate the outside integral

$$\begin{aligned}\bar{x} &= \frac{3}{8}\int_0^1 4x-4x^3\,dx \\ &= \frac{3}{8}\left[2x^2-x^4\Big|_0^1\right] \\ &= \frac{3}{8}[(2-1)-(0)] \\ &= \frac{3}{8}\end{aligned}$$

(c) Let's move on to $\bar{y}$

$$\begin{aligned}\bar{y} &= \frac{1}{m}\int\int_D y\rho(x,y)\,dA \\ &= \frac{3}{8}\int_0^1\int_0^{2-2x} y+3xy+y^2\,dy\,dx\end{aligned}$$

(d) Let's evaluate the inside integral

$$\begin{aligned}\int_0^{2-2x} y + 3xy + y^2\,dy &= \frac{1}{2}y^2 + \frac{3}{2}xy^2 + \frac{1}{3}y^3\Big|_0^{2-2x}\\ &= \frac{1}{2}(2-2x)^2 + \frac{3}{2}x(2-2x)^2 + \frac{1}{3}(2-2x)^3\\ &= \frac{10}{3}x^3 - 2x^2 - 6x + \frac{14}{3}\end{aligned}$$

(e) Evaluate the outside integral

$$\begin{aligned}\frac{3}{8}\int_0^1 \frac{10}{3}x^3 - 2x^2 - 6x + \frac{14}{3}\,dx &= \int_0^1 \frac{5}{4}x^3 - \frac{3}{4}x^2 - \frac{9}{4}x + \frac{7}{4}\,dx\\ &= \frac{5}{16}x^4 - \frac{1}{4}x^3 - \frac{9}{8}x + \frac{7}{4}\Big|_0^1\\ &= \frac{11}{16}\end{aligned}$$