MATH 232

CALCULUS III

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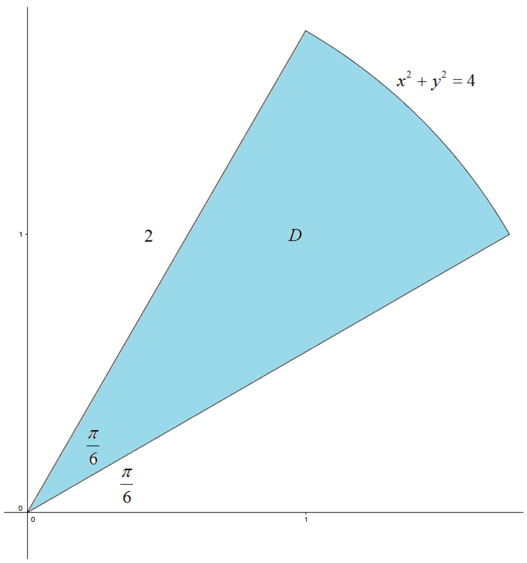
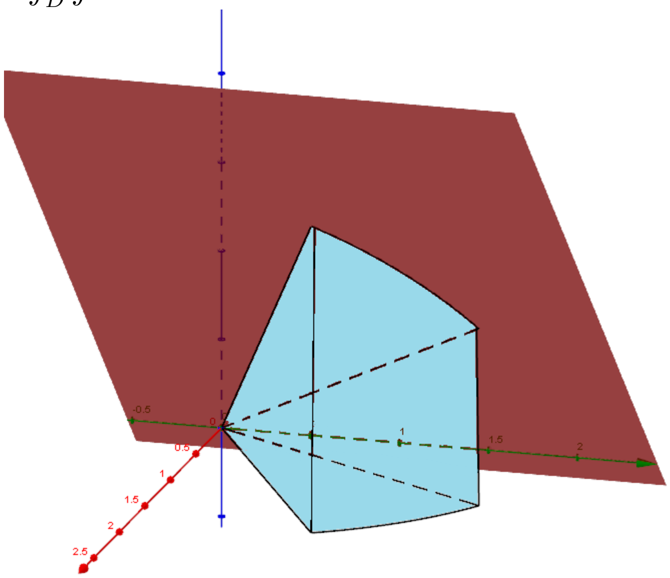
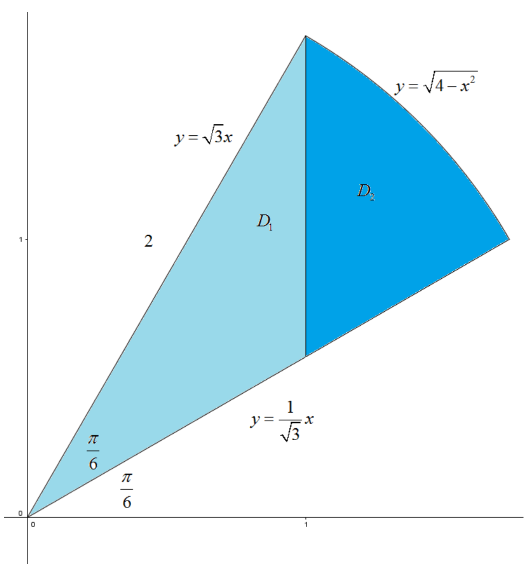
15.3 Double Integrals in Polar Coordinates

Suppose you want to evaluate the integral: $\int_D\int x\, dA$ over the region D (pictured below).





Notice that region D is neither vertically simple or horizontally simple. Because of this we need to split the region into two regions.



We can describe the two regions as

$$D_1 = \left\{(x,y)|0\le x\le 1,\ \frac{1}{\sqrt{3}}x\le y\le \sqrt{3}x\right\}$$

$$D_2 = \left\{(x,y)|1\le x\le \sqrt{3},\ \frac{1}{\sqrt{3}}x\le y\le \sqrt{4-x^2}\right\}$$

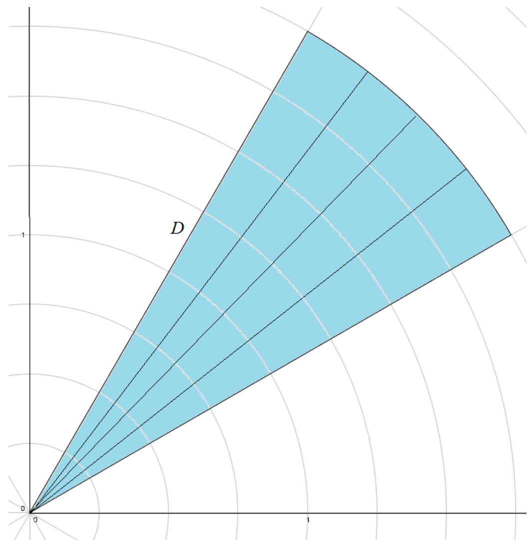
$$V=\int_{D_1}\int x\,dA+\int_{D_2}\int x\,dA$$

Setting up the integrals with their bounds we have the volume of the solid

$$V = \int_0^1 \int_{x/\sqrt{3}}^{\sqrt{3}x} x\ dy\ dx + \int_1^{\sqrt{3}} \int_{x/\sqrt{3}}^{\sqrt{4-x^2}} x\ dy\ dx$$

Even though this can be evaluated, we are going to find a better way of doing it. Can we describe the region D differently?

We can describe the region in polar coordinates very easily.

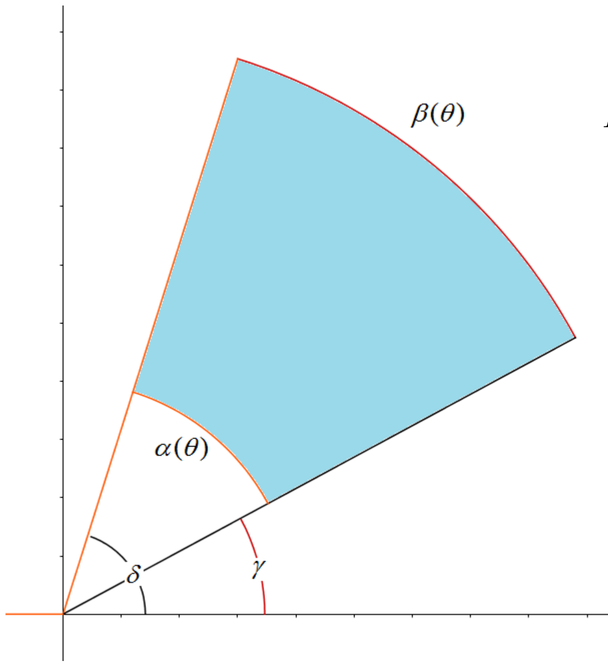


$$D = \left\{(r,\theta)|0 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\right\}$$

If we can convert our double into a polar double integral, then it might be easier to evaluate.

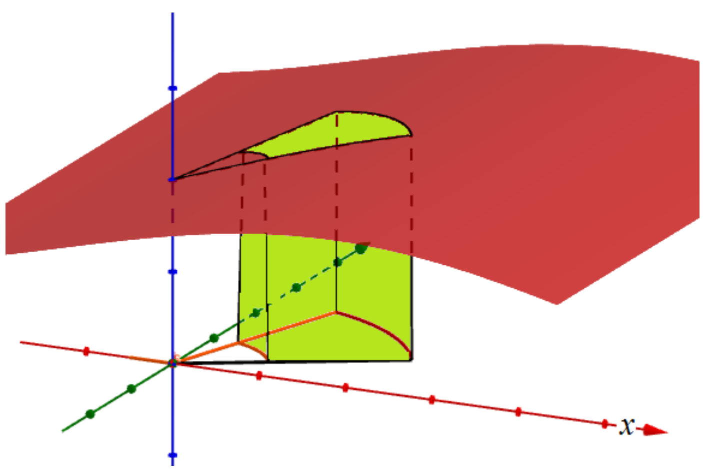
Definition 1: Radially Simple

A region that is radially simple is satisfies the following inequalities:



$$D = \{(r,\theta) \mid \gamma \leq \theta \leq \delta,\ \alpha(\theta) \leq r \leq \beta(\theta)\}$$

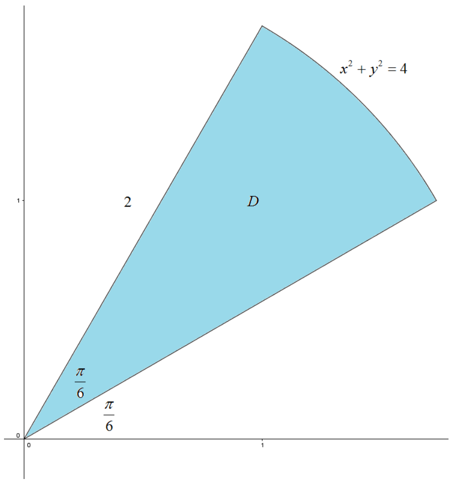
Definition 2: Polar Change for Double Integrals

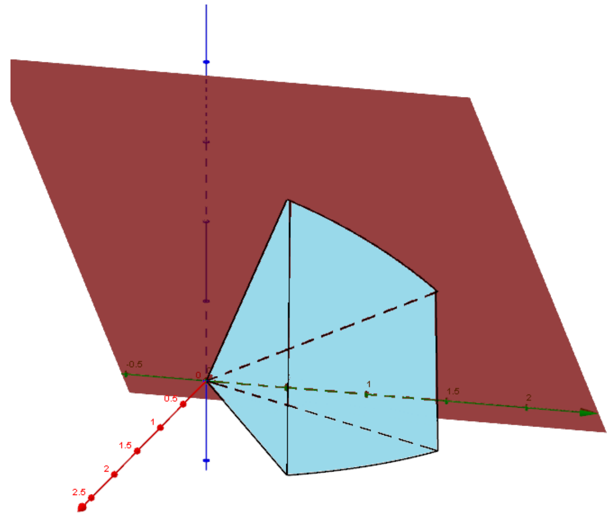


$$\int_D\int f(x,y)\,dA = \int_\gamma^\delta \int_{\alpha(\theta)}^{\beta(\theta)} f(r\cos(\theta), r\sin(\theta))\; r\; dr\; d\theta$$

Example 1

Let's get back to our first example. Evaluate $\displaystyle\int_D\int x\; dA$ over the region D.





We convert the region D to polar

$$D = \left\{(r, \theta) \mid \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3},\ 0 \leq r \leq 2\right\}$$

Using the polar change formula we get

$$\int_D\int x\; dA = \int_{\pi/6}^{\pi/3}\int_1^2 r\sin(\theta)\; r\; dr\; d\theta$$

1. Let's evaluate the inside integral

$$\int_1^2 r^2\cos(\theta)\; dr = \frac{1}{3}r^3\cos(\theta)\Big|_{r=0}^{r=2}$$

$$= \frac{8}{3}\cos(\theta)$$

2. Evaluate the outside integral

$$\int_{\pi/6}^{\pi/3}\frac{8}{3}\cos(\theta)\; d\theta = \frac{8}{3}\sin(\theta)\Big|_{\pi/6}^{\pi/3}$$

$$= \frac{8}{3}\sin(\pi/3) - \frac{8}{3}\sin(\pi/6)$$

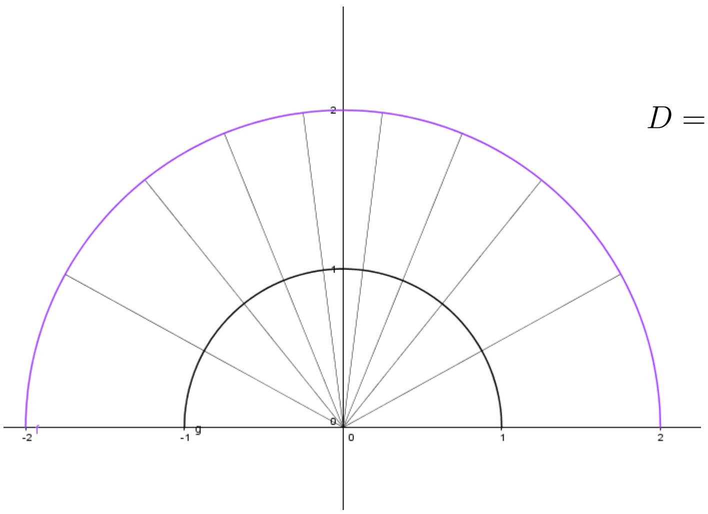
$$= \frac{8\sqrt{3}}{6} - \frac{8}{6}$$

$$= \frac{4}{3}\left(\sqrt{3} - 1\right)$$

Example 2

Evaluate $\int_D\int 3x+4y^2\, dA$ where D is the upper half of the plane bounded by $x^2+y^2 = 1$ and $x^2 + y^2 = 4$.

1. Start by sketching the region D



$D = \{(r, \theta) \mid 0 \le \theta \le \pi,\ 1 \le r \le 2\}$

2. Use the polar change formula

$$\int_D\int 3x + 4y^2\, dA = \int_0^\pi \int_1^2 \left(3r\cos\theta + 4r^2\sin^2\theta\right)\, r\, dr\, d\theta$$

$$= \int_0^\pi \int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta\, dr\, d\theta$$

(a) Let's evaluate the inside integral

$$\int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta\, dr = r^3\cos\theta + r^4\sin^2\theta\Big|_{r=1}^{r=2}$$

$$= 7\cos\theta + 15\sin^2\theta$$

(b) Evaluate the outside integral

$$\int_0^\pi 7\cos\theta + 15\sin^2\theta\, d\theta = \int_0^\pi 7\cos\theta + \frac{15}{2} - \frac{15}{2}\cos 2\theta\, d\theta$$

$$= 7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta\Big|_0^{\pi}$$

$$= \left[7\sin\pi + \frac{15\pi}{2} - \frac{15}{4}\sin 2\pi\right] - \left[7\sin 0 + \frac{15(0)}{2} - \frac{15}{4}\sin(0)\right]$$

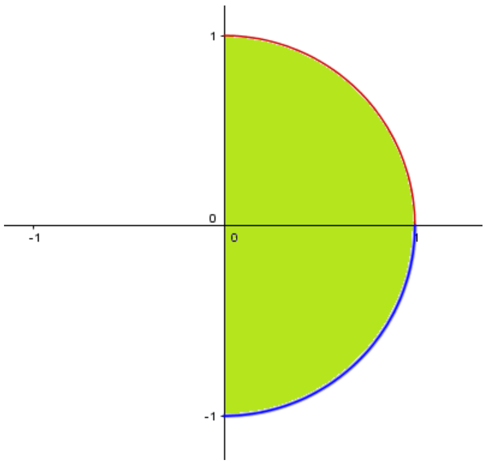
$$= \frac{15\pi}{2}$$

Example 3

Evaluate the iterated integral by first converting to polar

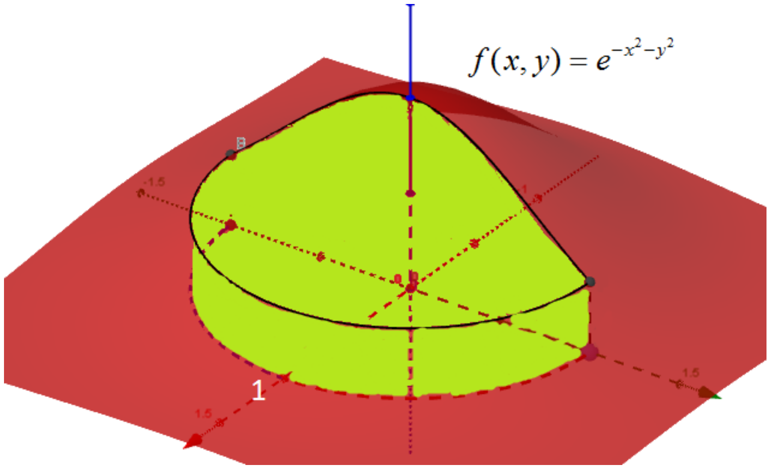
$$\int_0^2 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dy\, dx$$

Let's first sketch the region D



$$D = \{(r,\theta) \mid -\pi/2 \leq \theta\ \pi/2,\ 0 \leq r \leq 1\}$$

The graph below will give you an idea of what the integral represents.



Using the polar change formula the new integral is

$$\int_D\int e^{-x^2-y^2}\,dy\,dx = \int_{-\pi/2}^{\pi/2}\int_0^1 e^{-r^2}\,r\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^1 re^{-r^2}\,dr\,d\theta$$

1. Let's evaluate the inside integral

$$\int_0^1 re^{-r^2}\,dr$$

   (a) Let $u = -r^2$

   (b) $du = -2r\,dr \to -\frac{1}{2}\,du = r\,dr$

   (c) If $r = 0$, $u = 0$

   (d) If $r = 1$, $u = -1$

$$\begin{aligned}\int_0^1 re^{-r^2}\,dr &= \int_{-1}^{0} -\frac{1}{2}e^u\,du \\ &= -\frac{1}{2}e^u\Big|_{u=0}^{u=-1} \\ &= -\frac{1}{2}e^{-1} + \frac{1}{2}\end{aligned}$$

2. Evaluate the outside integral

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \frac{1}{2} - \frac{1}{2}e^{-1}\,d\theta &= \left(\frac{1}{2} - \frac{1}{2}e^{-1}\right)\theta\Big|_{-\pi/2}^{\pi/2} \\ &= \pi\left(\frac{1}{2} - \frac{1}{2}e^{-1}\right)\end{aligned}$$

Definition 3: Finding Area of Region Using Double Integrals

If $f(x,y) = 1$, then

$$\int_D\int 1\,dx\,dy = \text{ Area of } D$$

If you rewrite D so it's in polar, then

$$\int_\alpha^\beta\int_0^{h(\theta)} 1r\,dr\,d\theta$$