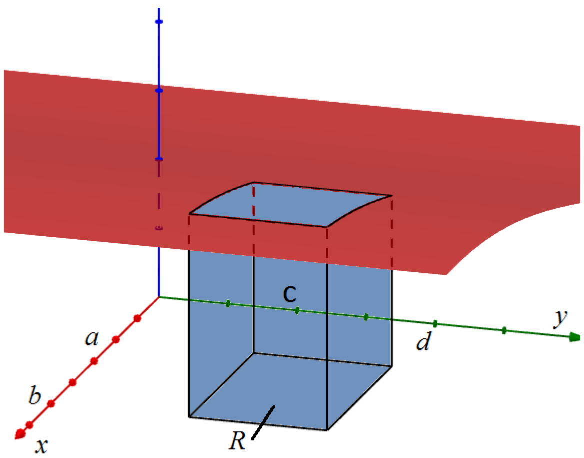
Math 232

Calculus III

Brian Veitch • Fall 2015 • Northern Illinois University

15.2 Double Integrals over General Regions

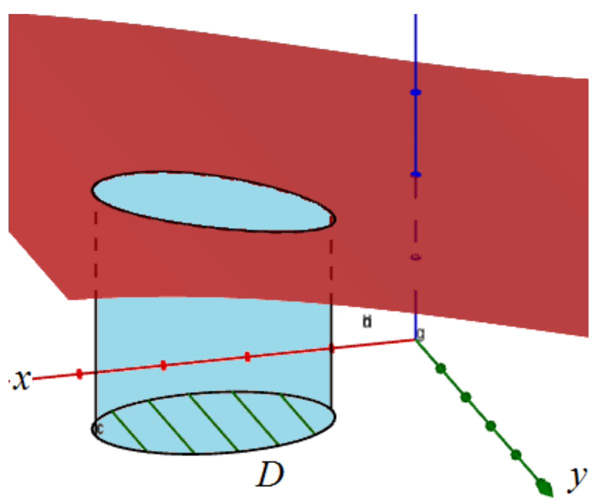
Recall that the volume under $f(x,y)$ over the rectangular region $R = [a,b] \times [c,d]$ is



$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

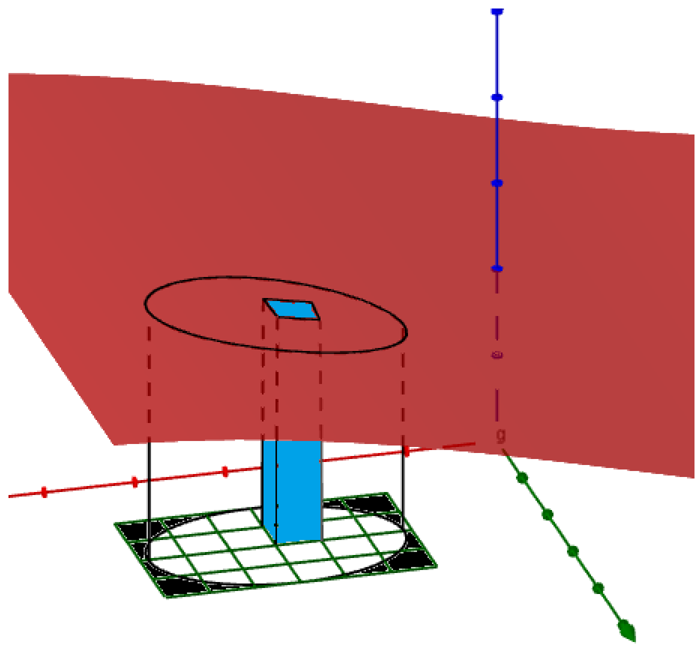
$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

What happens when the region R is not rectangular? What if it's a circular region? Triangular? Or just some odd shape?



The idea is very similar to the previous section. We define a new rectangular region R

that contains the region D. Once we have the rectangular region containing D we follow the same process as section 15.1.



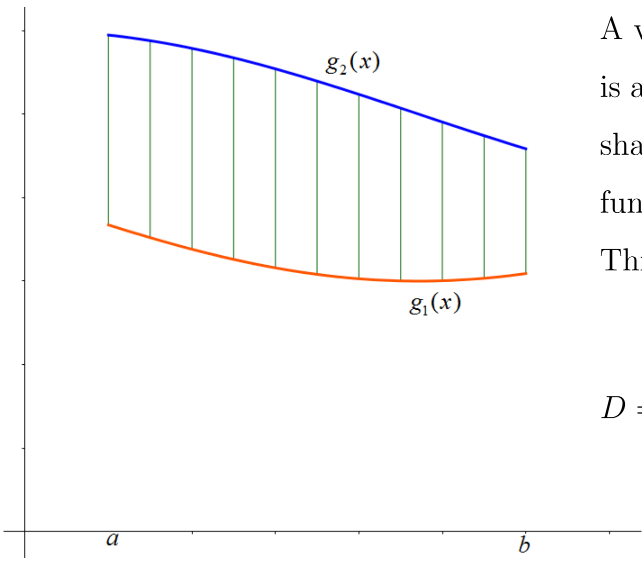
$$V = \int_D \int f(x,y)\, dA$$

Break the region into small rectangles. Find the volume of the rectangular box under $f(x,y)$. If any part of the rectangle is outside region D use $f(x_i, y_j) = 0$. That means the rectangular box has volume 0. I colored the rectangular regions we won't use in black. Now as you break the region into smaller and smaller rectangles, the rectangular regions we won't use get so small that they become insignificant.

Using the same logic as the previous section the volume under $f(x,y)$ can be found by

$$V = \int_D \int f(x,y)\, dA$$

Definition 1: Vertically Simple - Type 1

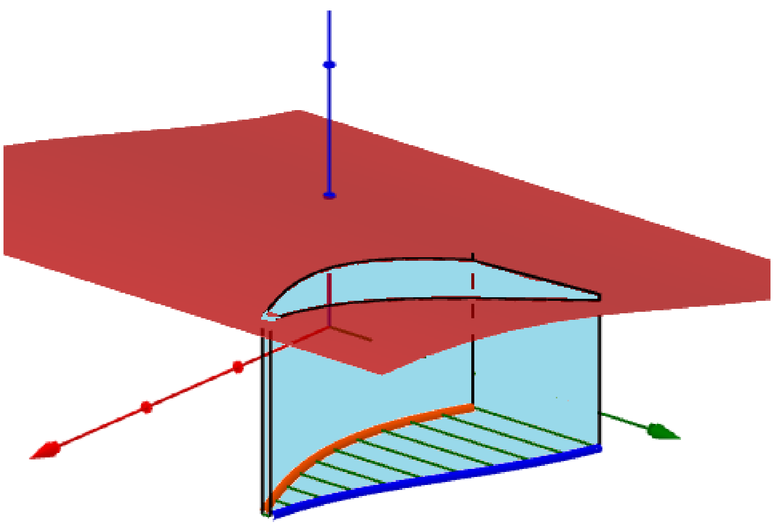


A vertically simple region (called Type 1) is a region where every vertical line drawn share the same upper function and bottom function $g_2(x)$ and $g_1(x)$

This region can be represented as such:

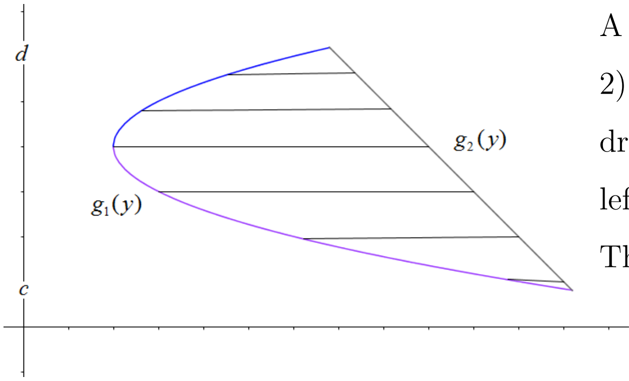
$$D = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

This let's us evaluate the volume by the following



$$V = \int_D \int f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Definition 2: Horizontally Simple - Type 2

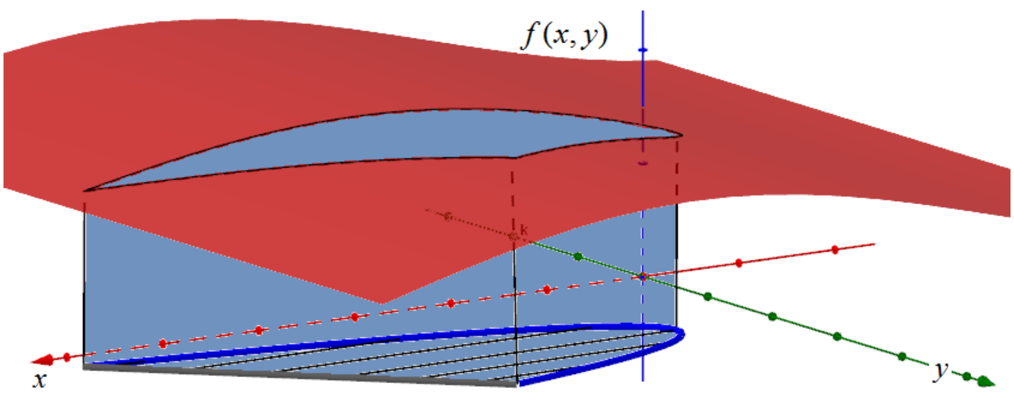


A horizontally simple region (called Type 2) is a region where every horizontal line drawn share the same right function and left function $g_2(y)$ and $g_1(y)$

This region can be represented as such:

$$D = \{(x, y)|c \leq y \leq d, g_1(y) \leq x \leq g_2(y)\}$$

This let's us evaluate the volume by the following

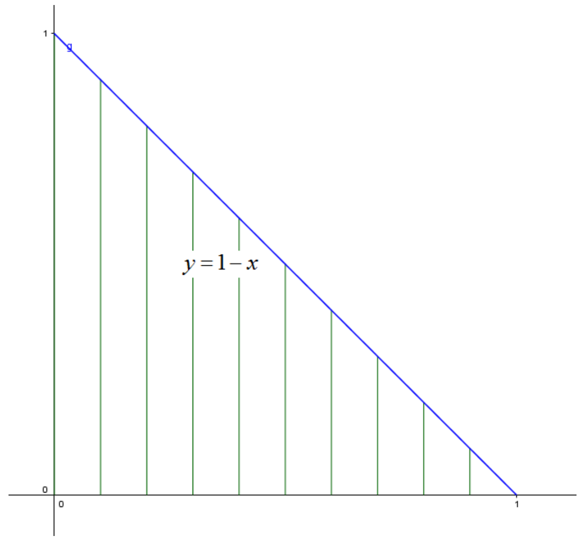


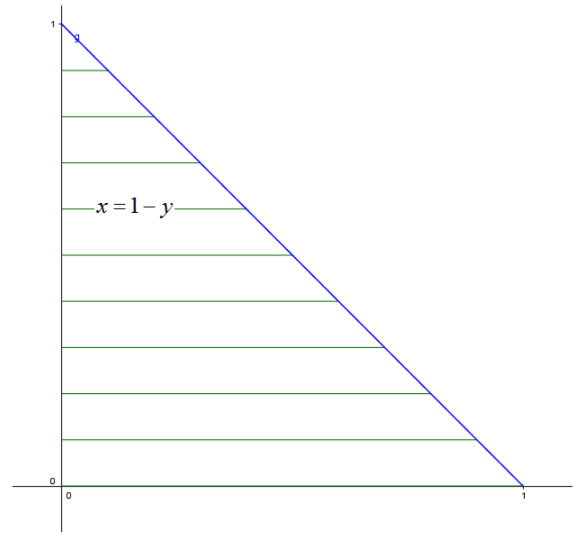
$$V = \int_D\int f(x, y)\, dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y)\, dx\, dy$$

Example 1

Evaluate $\int_D\int x + 2y + 1\, dA$ where is the region bounded by $y = 1 - x$, the x-axis, and the y-axis.

Let's sketch the region D to see if it's vertically or horizontally simple.





Notice that the region D is both vertically and horizontally simple. Let's evaluate the integral both ways.

1. D as a vertically simple region.

$$D = \{(x,y)|0 \le x \le 1,\ 0 \le y \le 1-x\}$$

$$\int\int_D x + 2y + 1\; dA = \int_0^1 \int_0^{x-1} x + 2y + 1\; dy\; dx$$

   (a) Let' evaluate the inside integral

$$\int_0^{1-x} x + 2y + 1\; dy = xy + y^2 + y\Big|_{y=0}^{y=1-x}$$

$$= \left[x(1-x) + (1-x)^2 + (1-x)\right] - \left[x(0) + (0)^2 + 0\right]$$

$$= x - x^2 + 1 - 2x + x^2 + 1 - x$$

$$= 2 - 2x$$

   (b) Evaluate the outside integral

$$\int_0^1 \int 2 - 2x\; dx = 2x - x^2\Big|_{x=0}^{x=1}$$

$$= \left[2(1) - (1)^2\right] - \left[2(0) - 0^2\right]$$

$$= 1$$

2. D as a horizontally simple region

$$D = \{(x, y) | 0 \leq y \leq 1,\ 0 \leq x \leq 1 - y\}$$

$$\int_D \int x + 2y + 1 \, dA = \int_0^1 \int_0^{1-y} x + 2y + 1 \, dx \, dy$$

(a) Evaluate the inside integral

$$\int_0^{1-y} x + 2y + 1 \, dx = \frac{1}{2}x^2 + 2xy + x \Big|_{x=0}^{x=1-y}$$

$$= \left[\frac{1}{2}(1-y)^2 + 2(1-y)y + (1-y)\right] - \left[\frac{1}{2}(0)^2 + 2(0)y + (0)\right]$$

$$= \frac{1}{2}(1 - 2y + y^2) + 2y - 2y^2 + 1 - y$$

$$= \frac{1}{2} - y + \frac{1}{2}y^2 + 2y - 2y^2 + 1 - y$$

$$= \frac{3}{2} - \frac{3}{2}y^2$$

(b) Evaluate the outside integral

$$\int_0^1 \frac{3}{2} - \frac{3}{2}y^2 \, dy = \frac{3}{2}y - \frac{1}{2}y^3 \Big|_{y=0}^{y=1}$$

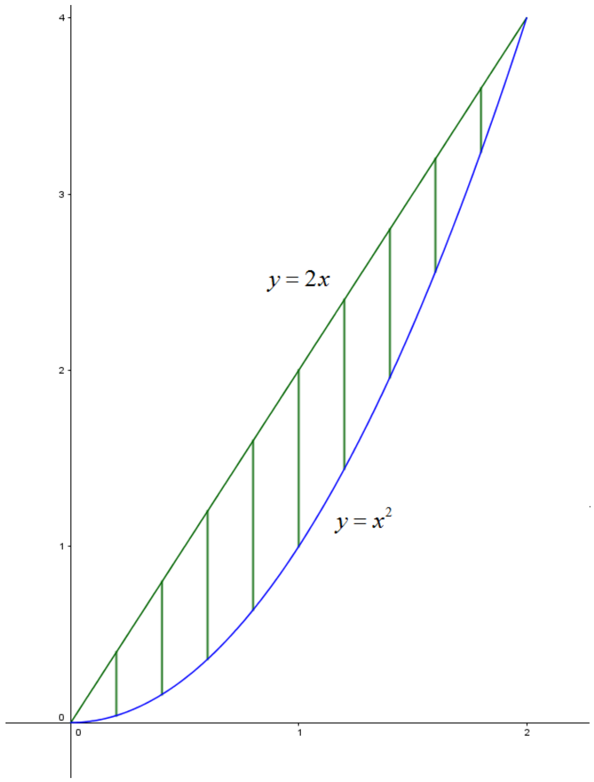
$$= \left[\frac{3}{2}(1) - \frac{1}{2}(1)^2\right] - \left[\frac{3}{2}(0) - \frac{1}{2}(0)^3\right]$$

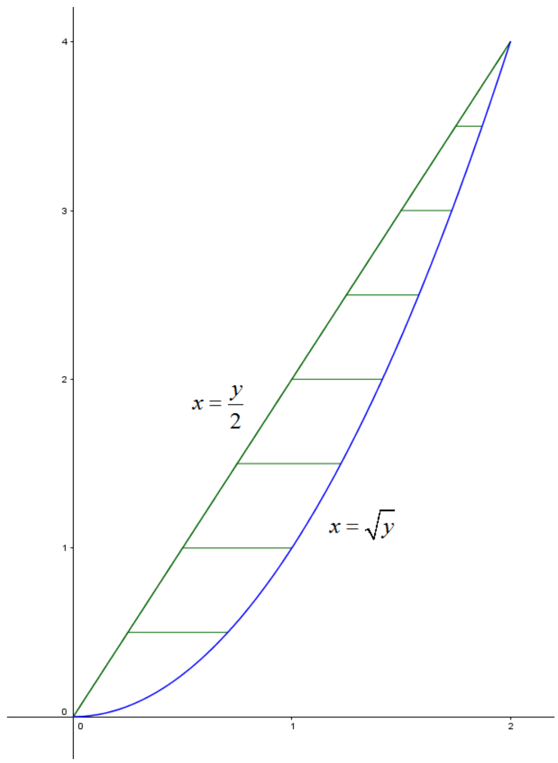
$$= 1$$

Example 2

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Like the previous problem, the region D is both vertically and horizontally simple.





$$\int_D\int x^2+y^2\;dA=\int_0^2\int_{x^2}^{2x}x^2+y^2\;dy\;dx$$

$$\int_D\int x^2+y^2\;dA=\int_0^4\int_{y/2}^{\sqrt{y}}x^2+y^2\;dx\;dy$$

1. D as a vertically simple region

   (a) Evaluate the inside integral

$$\int_{x^2}^{2x}x^2+y^2\;dy=x^2y+\frac{1}{3}y^3\Big|_{y=x^2}^{y=2x}$$

$$=\left[x^2(2x)+\frac{1}{3}(2x)^3\right]-\left[x^2(x^2)+\frac{1}{3}(x^2)^3\right]$$

$$=-\frac{1}{3}x^6-x^4+\frac{14}{3}x^3$$

   (b) Evaluate the outside integral

$$\int_0^2-\frac{1}{6}x^6-x^4+\frac{14}{3}x^3\;dx$$

$$=-\frac{1}{21}x^7-\frac{1}{5}x^5+\frac{14}{12}x^4\Big|_{x=0}^{x=2}$$

$$= \frac{216}{35}$$

2. D as a horizontally simple region

   (a) Evaluate the inside integral

$$\int_{y/2}^{\sqrt{y}} x^2 + y^2 \, dx = \frac{1}{3}x^3 + xy^2 \Big|_{x=y/2}^{x=\sqrt{y}}$$

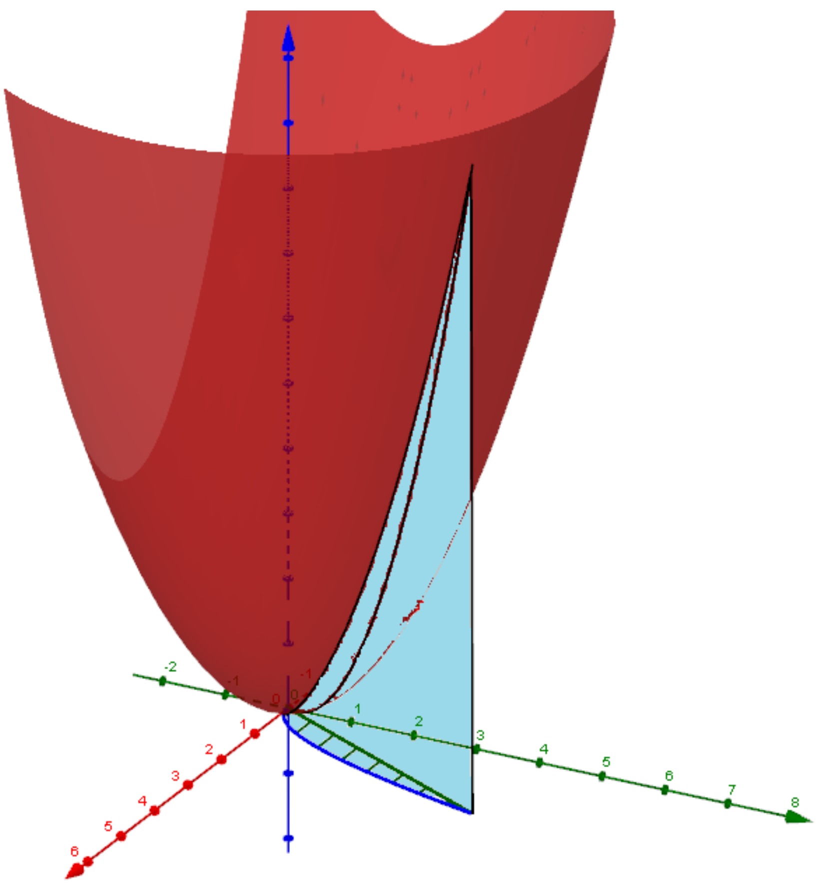
$$= \left[\frac{1}{3}(\sqrt{y})^3 + \sqrt{y}y^2\right] - \left[\frac{1}{3}(y/2)^3 + (y/2)y^2\right]$$

$$= y^{5/2} + \frac{1}{3}y^{3/2} - \frac{13}{24}y^3$$

   (b) Evaluate the outside integral

$$\int_0^4 y^{5/2} + \frac{1}{3}y^{3/2} - \frac{13}{24}y^3 \, dy = \frac{2}{7}y^{7/2} + \frac{2}{15}y^{5/2} - \frac{13}{96}y^4 \Big|_{y=0}^{y=4}$$

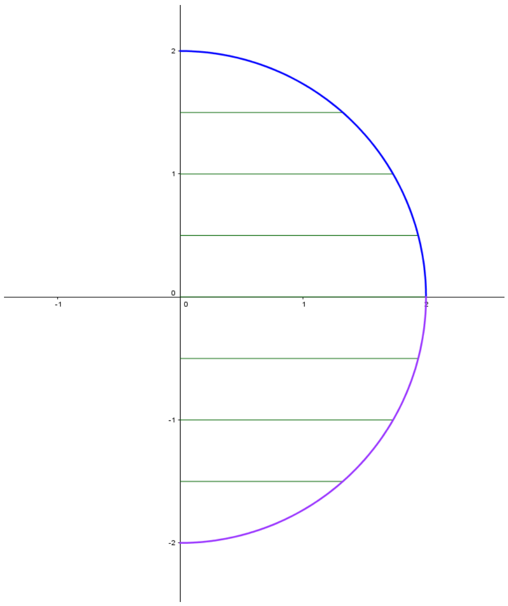
$$= \frac{216}{35}$$



Example 3

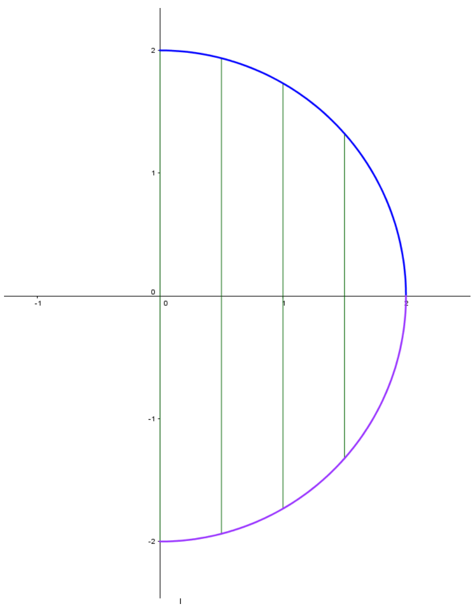
Sketch the region of the integration and change the order of integration.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$$



$$D = \left\{(x,y)| -2 \le y \le 2,\ 0 \le x \le \sqrt{4-y^2}\right\}$$

Switching the order you get



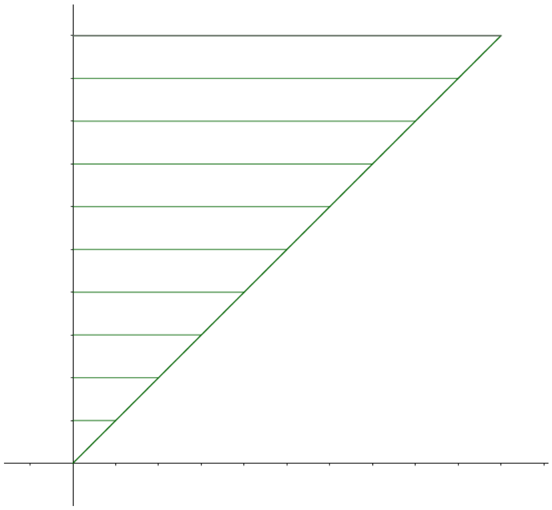
$$D = \left\{(x,y)|0 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\right\}$$

$$\int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

Example 4

Evaluate $\int_0^1 \int_x^1 \sin y^2 \, dy \, dx$ by reversing the order of integration.

We want to change the order of integration because we can't evaluate $\int \sin y^2 \, dy$.



Let $D = \{(x, y) | 0 \le x \le 1, \ x \le y \le 1\}$

Rewrite D as a horizontally simple region as

$$D = \{(x, y) | 0 \le y \le 1, \ 0 \le x \le y\}$$

$$\int_0^1 \int_0^y \sin y^2 \, dx \, dy$$

1. Evaluate the inside integral:

$$\int_0^y \sin y^2 \, dx = x \sin y^2 \Big|_{x=0}^{x=y}$$

$$= y \sin(y^2) - 0 \sin(y^2)$$

$$= y \sin(y^2)$$

2. Evaluate the outside integral:

$$\int_0^1 y \sin y^2 \, dy = -\frac{1}{2} \cos(y^2) \Big|_{y=0}^{y=1}$$

$$= -\frac{1}{2} \cos(1) + \frac{1}{2} \cos(0)$$

$$= \frac{1}{2} (1 - \cos 1)$$