Math 232

Calculus III

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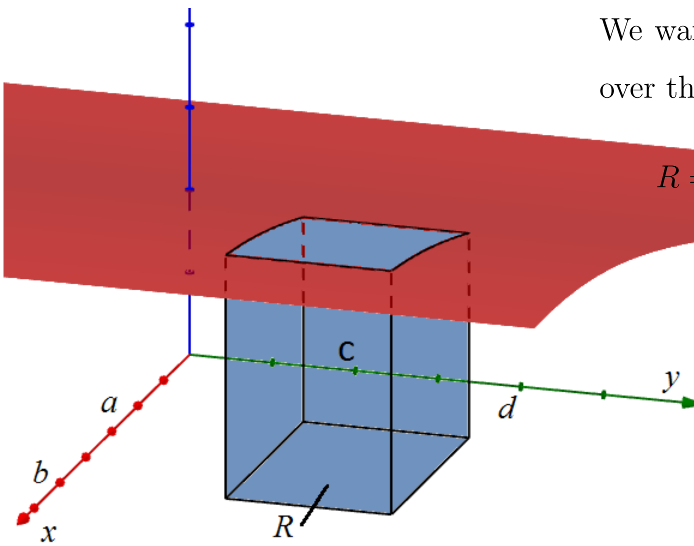
15.1 Double Integrals

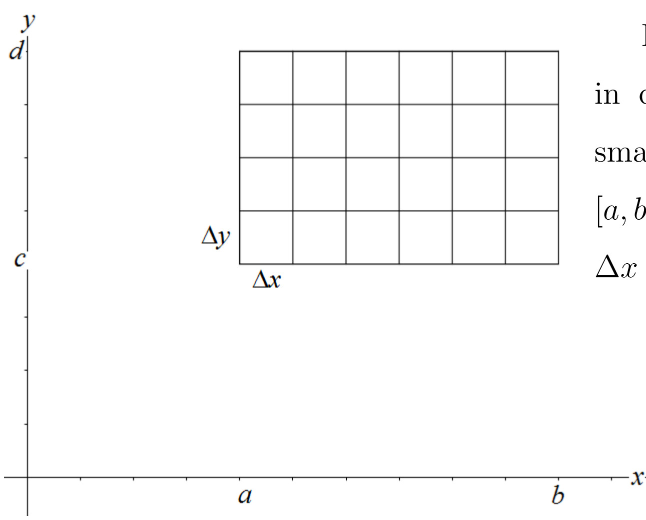
Definition 1: Integral of Function of Two Variables



We want the volume under the surface S over the rectangular region

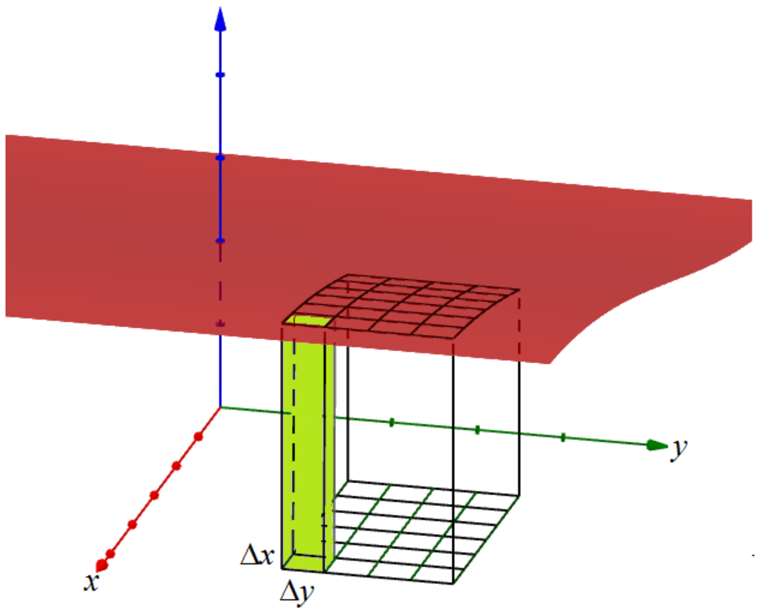
$$R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$$

$$V = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

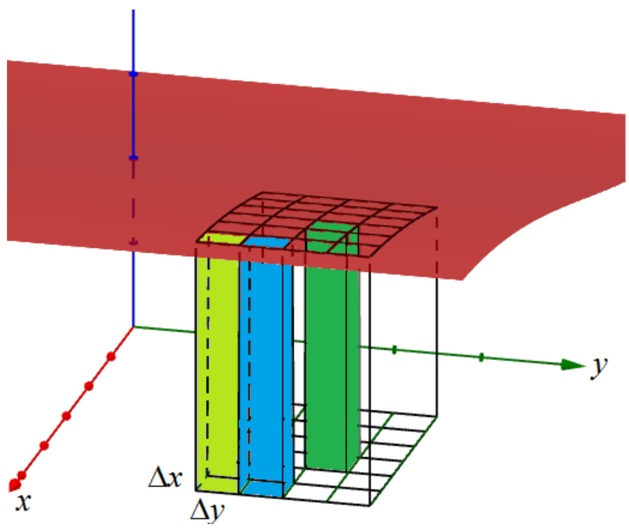


Let's take a look at R. Just like we did in calculus one, we split the domain into small rectangles. In this case the domain is $[a,b] \times [c,d]$ with each rectangle having width Δx and length Δy.

To find the volume under the surface we need to find the volume of the rectangular 'boxes' that lie underneath. Let's start with one rectangular box.



The area of the rectangular box is $f(x_1, y_2)\ \Delta x\ \Delta y$. Let's add some more rectangles.



The area of the group of rectangles is $f(x_1, y_1)\Delta x\ \Delta y + f(x_1, y_2)\Delta x\ \Delta y + f(x_3, y_3)\Delta x\ \Delta y$

Eventually we want to make infinitely many rectangles with $\Delta x \to 0$ and $\Delta y \to 0$.

$$\lim_{\# \to \infty} \sum f(x_i, y_j)\Delta x \Delta y = \int_R \int f(x, y)\, dA$$

Definition 2: Iterated Integrals - Fubini's Theorem

Let $f(x, y)$ be a function over $R = \{(x, y)|a \le x \le y, c \le y \le d\}$. Fubini's Theorem states that

$$\int_R \int f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Example 1

Evaluate $\int_1^2 \int_0^3 x^2 y\, dx\, dy$

1. Start with the inside integral: $\int_0^3 x^2 y\, dx$

$$\int_0^3 x^2 y\, dx = \frac{1}{3}x^3 y\Big|_{x=0}^{x=3}$$

$$= 9y - 0$$

$$= 9y$$

2. Plug $9y$ back into the main integral to get $\int_1^2 9y\, dy$.

$$\int_1^2 9y\, dy = \frac{9}{2}y^2\Big|_{y=1}^{y=2}$$

$$= 18 - \frac{9}{2}$$

$$= \frac{27}{2}$$

Example 2

Evaluate $\int_R \int xe^{xy}\, dA$ where $R = \{(x, y)|0 \le x \le 1, 1 \le yleq2\}$.

Since it doesn't matter which variable I start with first, I'm going to do the following

$$\int_1^2 \int_0^1 xe^{xy}\, dx\, dy$$

1. Let's start with the inner integral: $\int_0^1 xe^{xy}\,dx$

   We need to do by parts with this problem.

$$u = x \qquad dv = e^{xy}$$
$$du = dx \qquad v = \frac{1}{y}e^{xy}$$

$$\int xe^{xy}\,dx = \frac{x}{y}e^{xy} - \int \frac{1}{y}e^{xy}\,dx$$
$$\int_0^1 xe^{xy}\,dx = \frac{x}{y}e^{xy} - \frac{1}{y^2}e^{xy}\Big|_{x=0}^{x=1}$$
$$= \frac{1}{y}e^{y} - \frac{1}{y^2}e^{y} + \frac{1}{y^2}$$

2. Now we need to evaluate

$$\int_1^2 \frac{1}{y}e^{y} - \frac{1}{y^2}e^{y} + \frac{1}{y^2}\,dy$$

   Even though it can be done it's very annoying. It requires by parts a few more times. Now is a good time to check out Fubini's Theorem which states that we can change the order of integration.

$$\int_1^2\int_0^1 xe^{xy}\,dx\,dy = \int_0^1\int_1^2 xe^{xy}\,dy\,dx$$

   Let's see how the new order goes.

3. Let's start by evaluating the inside integral $\int_1^2 xe^{xy}\,dy$

$$\int_1^2 xe^{xy}\,dy = xe^{xy}\cdot\frac{1}{x}\Big|_{y=1}^{y=2}$$
$$= e^{xy}\big|_{y=1}^{y=2}$$
$$= e^{2x} - e^{x}$$

4. Next, we evaluate $\int_0^1 e^{2x} - e^{x}\,dx$

$$\int_0^1 e^{2x} - e^{x}\,dx = \frac{1}{2}e^{2x} - e^{x}\Big|_{x=0}^{x=1}$$

$$= \left[\frac{1}{2}e^2 - e^1\right] - \left[\frac{1}{2}e^0 - e^0\right]$$

$$= \frac{1}{2}e^2 - e + \frac{1}{2}$$

Using Fubini's Theorem helped quite a bit!

Definition 3: Special Case

If $f(x, y) = g(x)h(y)$ on $R = [a, b] \times [c, d]$ then

$$\int_R \int f(x, y)\, dA = \int \int g(x)h(y)\, dx\, dy = \int_c^d h(y)\, dy \cdot \int_a^b g(x)\, dx$$

Example 3

Evaluate $\int_R \int \sin(x)\cos(x)\, dA$ on $R = [0, \pi/4] \times [0, \pi/2]$.

$$\int_0^{\pi/2} \cos(y)\, dy \cdot \int_0^{\pi/4} \sin(x)\, dx$$

$$= \sin y|_0^{\pi/4} \cdot -\cos x|_0^{\pi/4}$$

$$= (1 - 0) \cdot \left(\frac{-\sqrt{2}}{2} + 1\right)$$

$$1 - \frac{\sqrt{2}}{2}$$