Math 232

Calculus III

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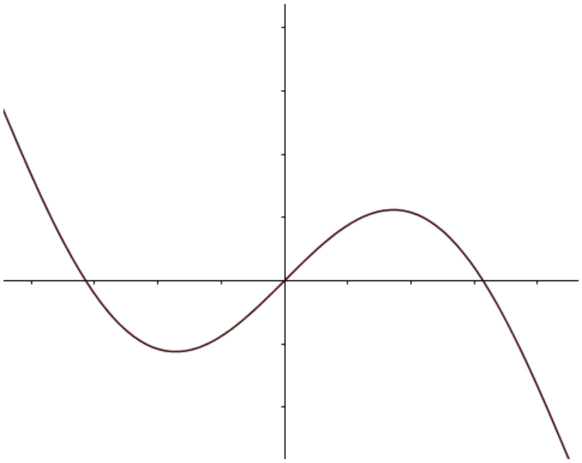
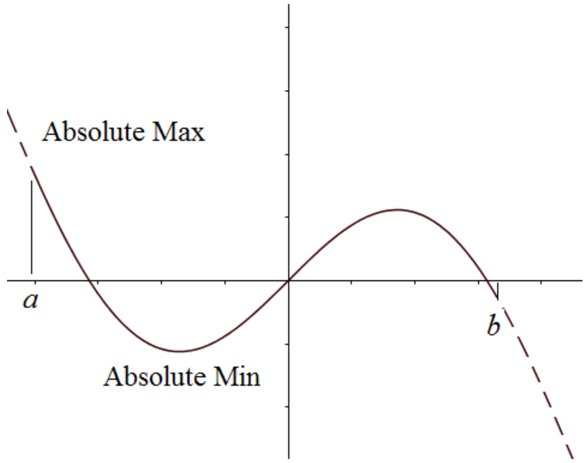
14.8 Lagrange Multipliers

This section deals with finding absolute extrema. Let's look back at calc I for a moment.

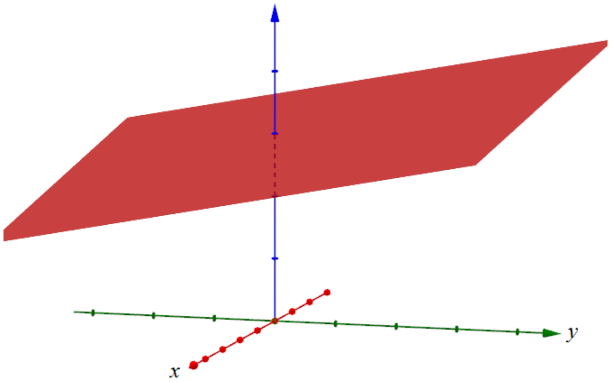


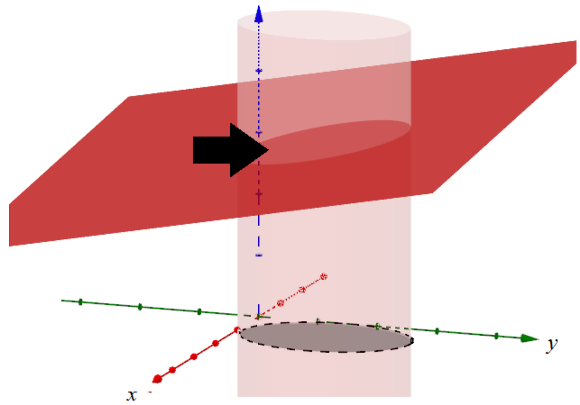


The graph on the left has an unrestricted domain. It has a local max and a local min but no absolute max or min. When the domain is not restricted we are not guaranteed absolute extrema. In Calc I we had the **Extreme Value Theorem** that stated that if f is continuous on a closed interval, f must have an absolute max and an absolute min.

The graph on the right has a restricted (closed) domain $[a, b]$. Because of this f is guaranteed absolute extrema. It just so happens the absolute max occured at an endpoint.

Let's consider now what happens for a function $f(x, y)$ in three dimensional space.





The graph on the left is a plane on an unrestricted domain. When it moves to the right it just keeps going up (no absolute max). When it moves to the left it keeps going down (no absolute min).

The graph on the right has a restricted domain. It's now possible that f might have absolute extrema. Note that the restriction domain is a function of x and y.

Method of Lagrange Multipliers

Steps 1: Method of Lagrange Multipliers

Find the absolute max and min of $f(x,y)$ subject to the contraints $g(x,y)=k$ provided $\nabla g \neq 0$.

1. Find all x and y such that

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y), & \lambda \text{ is Lagrante Multiplier} \\ g(x,y) = k \end{cases}$$

2. Then evaluate $f(x,y)$ at all points (x,y) found above.

3. The largest of these values is the absolute max.

4. The smallest of these values is the absolute min.

Example 1

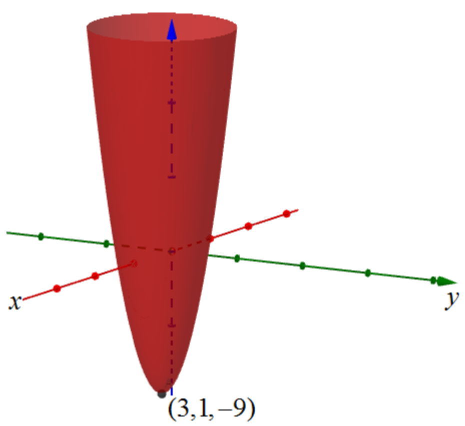
Find the minimum to $f(x,y) = x^2 + y^2 - 6x - 2y + 1$.

Without any restrictions, we follow the method from 14.7.

1. Find $\nabla f(x,y)$
$$\nabla f(x,y) = < 2x-6, 2y-2 >$$
.

2. Solve $f_x = 0$ and $f_y = 0$
$$2x-6=0 \Rightarrow x=3$$
$$2y-2=0 \Rightarrow y=1$$

3. A point (x,y) that works for both equations is $(3,1)$.



Looking at the graph of $f(x,y)$ we see that (without any restrictions) the minimum is at $(3,1,-9)$

Example 2

Find the absolute minimum of $f(x,y) = x^2+y^2-6x-2y+1$ subject to the constraint $x+y=2$

We have to use the Method of Lagrange Multipliers. Let's make our constraint function $g(x,y) = x+y$

1. Let's start with our overall objective. We need to solve
$$\begin{cases} \nabla f(x,y) & = \lambda \nabla g(x) \\ g(x,y) & = 2 \end{cases}$$

2. Find ∇f, and λg

$$\nabla f = < 2x - 6, 2y - 2 >$$

$$\nabla g = < 1, 1 >$$

3. Solve

$$\begin{cases} < 2x - 6, 2y - 2 > = \lambda < 1, 1 > \\ x + y = 2 \end{cases}$$

by solving

$$2x - 6 = 1\lambda$$

$$2y - 2 = 1\lambda$$

$$x + y = 2$$

4. One of the first attempts at solving these problems is to solve one equation for λ and plug it into the other equation.

$$2x - 6 = \lambda$$

Plug $\lambda = 2x - 6$ into $2y - 2 = \lambda$

$$2y - 2 = 1(2x - 6) \Rightarrow 2x - 2y = 4$$

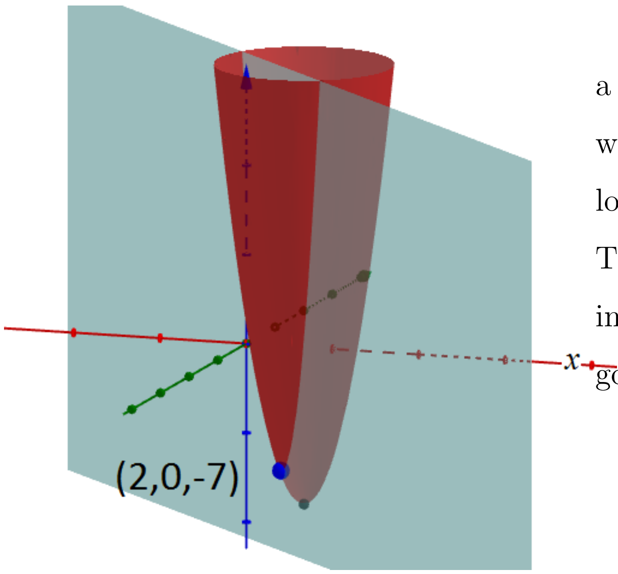
This doesn't give us solutions for x and y. Now we have to solve the system

$$2x - 2y = 4$$

$$x + y = 2$$

Doing this gives us the solution $x = 2$, $y = 0$.

5. Is $(2, 0, -7)$ the minimum? Let's look at the graph.



Remember that the absolute min must be a point on the plane $x + y = 2$. You can see where the plane and $f(x, y)$ intersect. The lowest point on f appears to be $(2, 0, -7)$. The graph has no absolute max because the intersections between f and the plane keep going up.

Example 3

Find the extreme values of $f(x, y) = x^2+y^2-6x-2y+1$ subject to $(x-1)^2+(y-1)^2 = 1$.

Let the constraint function be $g(x, y) = (x - 1)^2 + (y - 1)^2$

1. $\nabla f =< 2x - 6, 2y - 2 >$
2. $\nabla g =< 2(x - 1), 2(y - 1) >$
3. Solve

$$\begin{cases} < 2x - 6, 2y - 2 >= \lambda < 2x - 2, 2y - 2 > \\ (x - 1)^2 + (y - 1)^2 = 1 \end{cases}$$

$$\begin{cases} 2x - 6 = \lambda(2x - 2) \\ 2y - 2 = \lambda(2y - 2) \\ (x - 1)^2 + (y - 1)^2 = 1 \end{cases}$$

4. Solve one equation for λ $\lambda = \dfrac{2x - 6}{2x - 2}$
5. Plug λ into $2y - 2 = \lambda(2y - 2)$

$$2(y - 1) = \frac{2x - 6}{2x - 2}(2y - 2)$$

$$y - 1 = \frac{x-3}{x-1}(y-1)$$

This equation can only be true if $y = 1$.

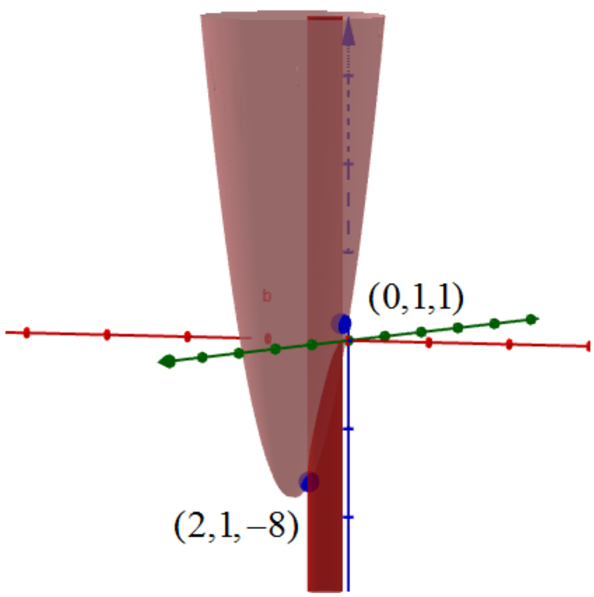
6. Solve for x by plugging $y = 1$ into $(x-1)^2 + (y-1)^2 = 1$

$$(x-1)^2 + 0^2 = 1$$

$$(x-1)^2 = 1$$

$$x = 2, 0$$

7. Plug $(2, 1)$ and $(0, 1)$ into $f(x, y)$ to see which is the largest and which is the smallest.



$f(2, 1) = -8$ Minimum

$f(0, 1) = 1$ Maximum