Math 232

Calculus III

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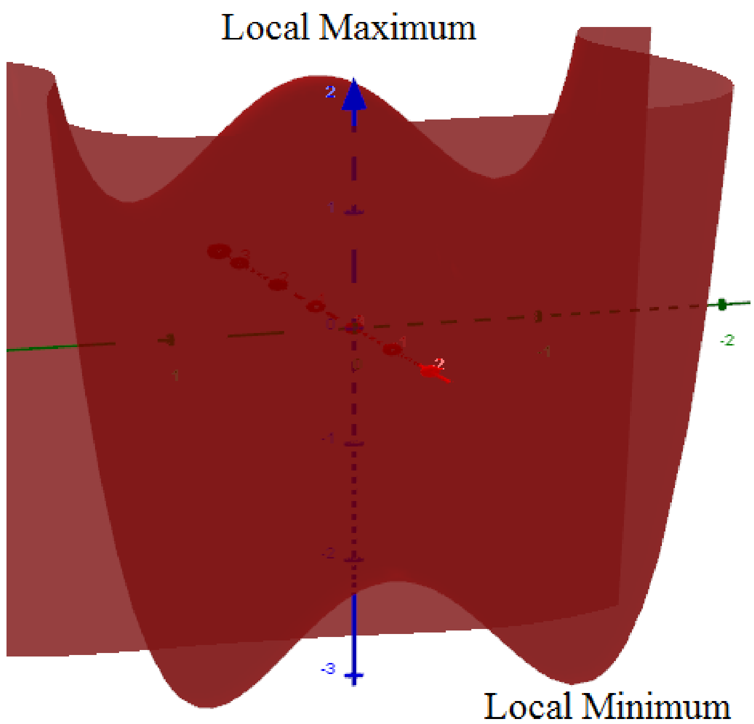
14.7 Maximum and Minimum Values

Definition 1: Local Extrema

$f(x,y)$ has a local minimum at (a,b) if $f(x,y) \geq f(a,b)$ when (x,y) is near (a,b).

$f(x,y)$ has a local maximum at (a,b) if $f(x,y) \leq f(a,b)$ when (x,y) is near (a,b).



Theorem 1

If f has a local maximum or minimum at (a, b) and f_x and f_y exist, then $f_x(a, b) = 0$ AND $f_y(a, b) = 0$

Like in calculus I, it means all **potential max/mins** must occur when $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Call these **Critical Points**.

Example 1

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Find all critical points.

1. Start by finding f_x and f_y.

$$f_x = 2x - 2$$

$$f_y = 2y - 6$$

2. Find all points (x, y) where $f_x = 0$ and $f_y = 0$.

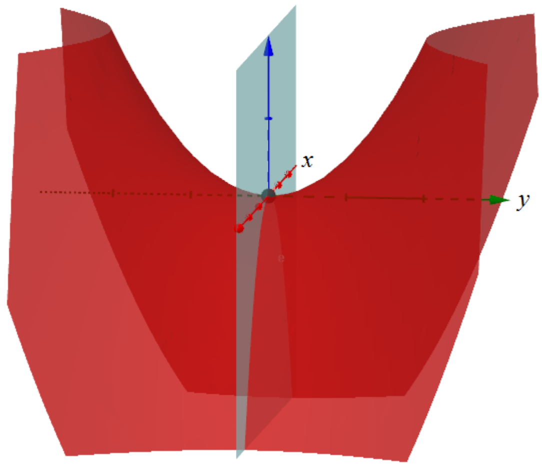
$$f_x = 2x - 2 = 0 \text{ when } x = 1$$

$$f_y = 2y - 6 = 0 \text{ when } y = 3$$

3. Since there are no contradictions between our solutions, we have a critical point at $(1, 3)$.

In Calculus I we talked about how not all critical points are local extrema. We can extend that idea to the three-dimensional coordinate system. Consider the graph below.

Definition 2: Saddle Point at (a, b)



Consider $P(a, b)$ at the origin. If you travel the curve along the y-axis (in either direction) the slope is positive. As you travel along the x-axis (in either direction) the slope is negative. The point can neither be a local maximum nor a local minimum.

We call the point (a, b) a **saddle point**.

Definition 3: Second Derivative Test

Assume the second partial derivatives of f are continuous on a disk with center (a, b). Suppose $f_x(a, b) = 0$, $f_y(a, b) = 0$ and

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

1. If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum.
3. If $D < 0$, then $f(a, b)$ is neither a maximum nor minimum (Saddle Point)
4. If $D = 0$, the test is inconclusive.

Example 2

Find the local extrema and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

1. Let's start by finding all the critical points.

   (a) $f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$. We must solve

$$f_x = 4x^3 - 4y = 0$$

$$f_y = 4y^3 - 4x = 0$$

Solving the first equation for y we get $y = x^3$. Plug this into the second equation and we get

$$4(x^3)^3 - 4x = 0$$

$$4x^9 - 4x = 0$$

$$4x(x^8 - 1) = 0$$

$$4x(x^4 - 1)(x^4 + 1) = 0$$

$$4x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$$

$$4x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1) = 0$$

$$x = 0, 1, -1$$

(b) For each x-value we need its corresponding y-value.

$$\text{If } x = 0, y = (0)^3 = 0$$

$$\text{If } x = -1, y = (-1)^3 = -1$$

$$\text{If } x = 1, y = (1)^3 = 1$$

(c) We have three critical points: $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

2. For each of the critical points we need to find $D(x, y)$. To do this we need f_{xx}, f_{yy}, and f_{xy}.

$$f_{xx} = 12x^2$$

$$f_{yy} = 12y^2$$

$$f_{xy} = -4$$

(a) Critical Point $(0,0)$:

$$\begin{aligned} D(0,0) &= f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2 \\ &= 12(0)^2 \cdot 12(0)^2 - [-4]^2 \\ &= -16 \end{aligned}$$

Since $D < 0$, the critical point $(0,0,1)$ is a saddle point.

(b) Critical Point $(1,1)$:

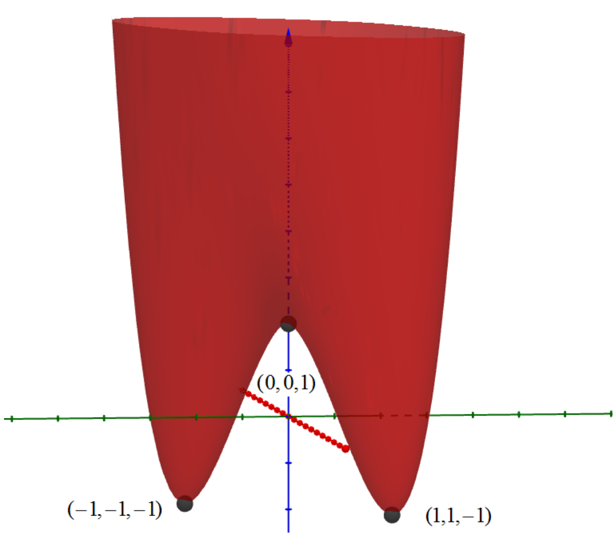
$$\begin{aligned} D(1,1) &= f_{xx}(1,1)f_{yy}(1,1) - [f_{xy}(1,1)]^2 \\ &= 12(1)^2 \cdot 12(1)^2 - [-4]^2 \\ & 128 \end{aligned}$$

Since $D > 0$ and $f_{xx}(1,1) > 0$, the critical point $(1,1,-1)$ is a local minimum.

(c) Critical Point $(-1,-1)$:

$$\begin{aligned} D(-1,-1) &= f_{xx}(-1,-1)f_{yy}(-1,-1) - [f_{xy}(-1,-1)]^2 \\ &= 12(-1)^2 \cdot 12(-1)^2 - [-4]^2 \\ & 128 \end{aligned}$$

Since $D > 0$ and $f_{xx}(-1,-1) > 0$, the critical point $(-1,-1,-1)$ is a local minimum.



Example 3

Find the shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$

The distance between two points (x, y, z) and $(1, 0, -2)$ in a three-dimensional coordinates system is

$$d = \sqrt{(x-1)^2 + (y-0)^2 + (z+2)^2}$$

One problem we have is the function $d(x, y)$ has a z in it. We need to write z in term of x and y. Since we know the point must be on the plane $z = 4 - x - 2y$ we can substitute this in for z

$$d = \sqrt{(x-1)^2 + y^2 + ((4 - x - 2y) + 2)}$$

$$d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}$$

1. Find d_x and d_y

$$d_x = \frac{2(x-1) + 2(6-x-2y)\cdot(-1)}{2\sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}}$$
$$d_x = \frac{-14 + 4x + 4y}{2\sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}}$$

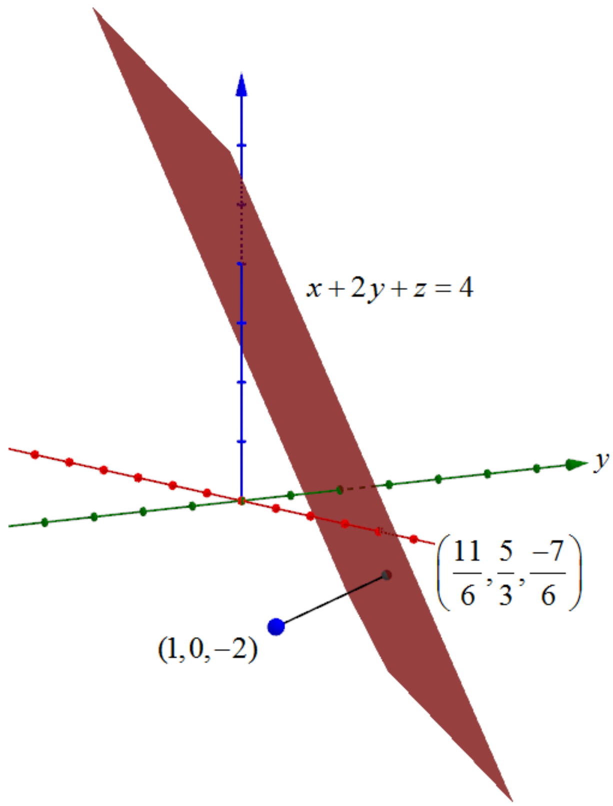
$$d_y = \frac{2y + 2(6-x-2y)\cdot(-2)}{2\sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}}$$
$$d_y = \frac{-24 + 4x + 10y}{2\sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}}$$

2. Solve $d_x = 0$ and $d_y = 0$

$$d_x = 0 \Rightarrow 4x + 4y - 14 = 0$$

$$d_y = 0 \Rightarrow 4x + 10y - 24 = 0$$

Solving these two equations we get $y = \frac{5}{3}$ and $x = \frac{11}{6}$


x + 2y + z = 4
y
(11/6, 5/3, −7/6)
(1,0,−2)