Math 232

Calculus III

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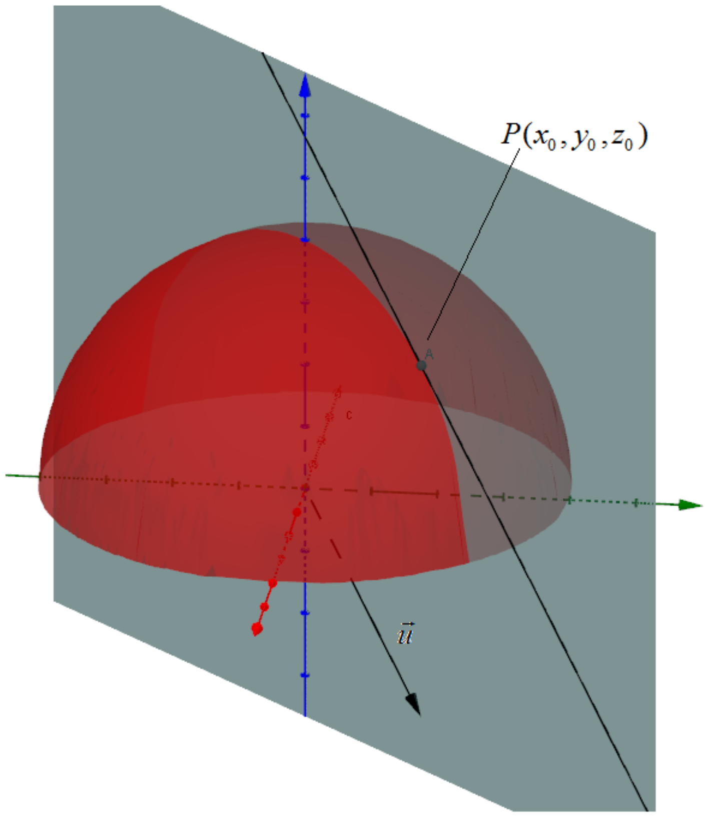
14.6 Directional Derivatives and the Gradient Vector

In previous sections we discussed partial derivatives. With partial derivatives we can find the slope of the tangent line in the direction of i (slope parallel to x-axis) and in the direction of j (slope parallel to the y-axis). In this section we find the way to determine the rate of change of z in any direction.

Definition 1: Directional Derivative

If f is differentiable in x and y then f has a directional derivative in the direction of any unit vector $\vec{u} =< a, b >$ and

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$



The directional derivative will give us the slope of the tangent line T to the curve at the point $P(x_0, y_0, z_0)$ in the direction of $\vec{u}$.

Note: If you're just looking for the change in x, then $\vec{u} =< 1, 0 >$. This means

$$D_{\vec{u}}f(x, y) = f_x(x, y)$$

which is exactly what we would expect.

Example 1

Let $f(x, y) = \sqrt{4 - x^2 - y^2}$. Find the directional derivative $D_{\vec{u}}f(x, y)$ at $P(1, 1)$ in the direction of $\vec{v} =< 1, 1 >$.

This example is exactly the graph shown above so we already know what this should look like.

1. $f_x(x,y) = \dfrac{-x}{\sqrt{4-x^2-y^2}}, \qquad f_x(1,1) = \dfrac{-1}{\sqrt{4-1^2-1^2}} = -\dfrac{1}{\sqrt{2}}$
2. $f_y(x,y) = \dfrac{-y}{\sqrt{4-x^2-y^2}}, \qquad f_y(1,1) = \dfrac{-1}{\sqrt{4-1^2-1^2}} = -\dfrac{1}{\sqrt{2}}$
3. $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{<1,1>}{\sqrt{2}} =< \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} >=< a,b >$

$$\begin{aligned}
D_{\vec{u}}f(x,y) &= f_x(x,y)a + f_y(x,y)b \\
D_{\vec{u}}f(x,y) &= \frac{-ax}{\sqrt{4-x^2-y^2}} + \frac{-by}{\sqrt{4-x^2-y^2}} \\
D_{\vec{u}}f(1,1) &= -\frac{-(1/\sqrt{2})(1))}{\sqrt{4-1^2-1^2}} + \frac{-(1/\sqrt{2})(1)}{\sqrt{4-1^2-1^2}} \\
D_{\vec{u}}f(1,1) &= -1
\end{aligned}$$

The rate of change in z in the direction of $\vec{u} =< \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} >$ is -1.

Not only do we know the slope, we can find the equation of the tangent line at $P(1,1)$.

1. Direction: $< \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, -1 >$
2. Point: $P(x_0, y_0, z_0) = (1, 1, \sqrt{2})$

$$T =< 1, 1, \sqrt{2} > +t < \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1 >$$

$$x = 1 + \frac{t}{\sqrt{2}}, \;\; y = 1 + \frac{t}{\sqrt{2}}, \;\; z = \sqrt{2} - t$$

Let's go back to the directional derivative formula

$$D_{\vec{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$$

can also be written

$$D_{\vec{u}}f(x,y) =< f_x(x,y), f_y(x,y) > \cdot < a,b >$$

Since $< f_x(x,y), f_y(x,y) >$ shows up more in the textbook we will give it a special name.

Definition 2: The Gradient of f, ∇f

Given a function of two variables x and y, then the gradient of f is

$$\nabla f = \nabla f(x,y) =< f_x(x,y), f_y(x,y) >$$

Definition 3: Directional Derivative of f

Using the gradient ∇f we can define the directional derivative to be

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

Example 2

Let $f(x,y) = \sin x + e^{xy}$. Find the gradient ∇f

We need to find f_x and f_y

1. $f_x(x,y) = \cos(x) + ye^{xy}$
2. $f_y(x,y) = xe^{xy}$

$$\nabla f(x,y) =< \cos(x) + ye^{xy}, xe^{xy} >$$

Example 3

Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ in the direction of $\vec{v} = 2i + 5j$.

Recall the directional derivative formula is $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$. We need to find $\vec{u}$, f_x, $f_x(2,-1)$, f_y, and $f_y(2,-1)$.

1. We are given $\vec{v} =< 2, 5 >$. This is not a unit vector. We need to convert it to one

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{< 2, 5 >}{\sqrt{29}} =< \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} >$$

2. Next we need $\nabla f(x,y)$

$$\nabla f(x,y) =< f_x(x,y), f_y(x,y)$$

(a) $f_x(x, y) = 2xy^3$

(b) $f_y(x, y) = 3x^2y^2 - 4$

$$\nabla f(x, y) = < 2xy^3, 3x^2y^2 - 4 >$$

3. Now we need $\nabla f(2, -1)$

$$\nabla f(2, -1) = < 2(2)(-1)^3, 3(2)^2(-1)^2 - 4 >= < -4, 8 >$$

4. Finally let's calculate $D_{\vec{u}} f(2, -1)$

$$\begin{aligned} D_{\vec{u}} f(2, -1) &= \nabla f(2, -1) \cdot < \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} > \\ &= < -4, 8 > \cdot < \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} > \\ &= -\frac{8}{\sqrt{29}} + \frac{40}{\sqrt{29}} \\ &= \frac{32}{\sqrt{29}} \end{aligned}$$

I will most likely show you an interactive graph of this in class. It will be easier than drawing it by hand.

Definition 4: The Gradient of f with 3 Variables

Let Let $f(x, y, z)$ be a function of three variables x, y, and z. The gradient of f is

$$\nabla f(x, y, z) = < f_x, f_y, f_z >= \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Maximizing the Directional Derivative

Of all the possible directional deriatives, which DIRECTION has the fastest positive change and what is that max change?

Theorem 1

Suppose f is a differentiable function of 2 or 3 variables. The max value of the directional derivative $D_{\vec{u}} f = |\nabla f|$ and it's in the direction of the gradient vector of ∇f.

Example 4

If $f(x, y) = xe^y$,

1. find the rate of change of f at $(2, 0)$ in the direction of $\vec{u} =< \frac{-3}{5}, \frac{4}{5} >$.
2. In what direction does f have the max rate of change? What is it?

1. This question is simplify looking for the rate of change at $(2, 0)$ in the direction of $\vec{u}$.

$$D_{\vec{u}}f(2,0) = \nabla f(2,0) \cdot < \frac{-3}{5}, \frac{4}{5} >$$

   (a) $\nabla f(x, y) =< e^y, xe^y >$

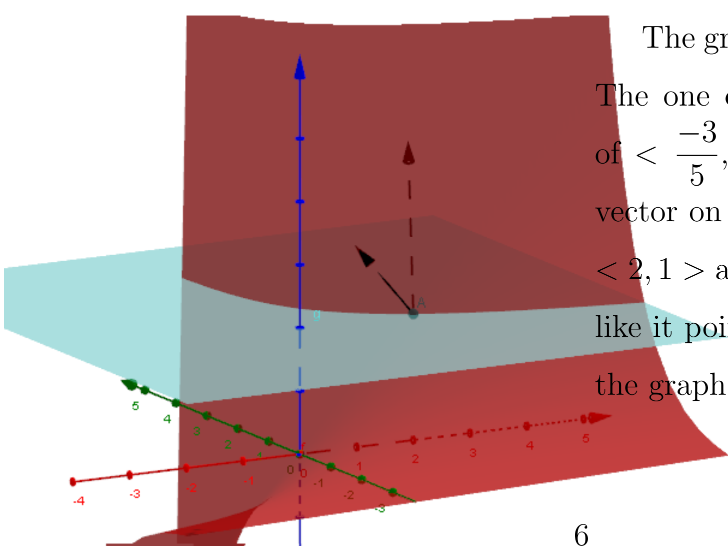
   (b) $\nabla f(2, 0) =< e^0, 2e^0 >=< 1, 2 >$

$$\begin{aligned} D_{\vec{u}}f(2,0) &=< 1,2 > \cdot < -\frac{3}{5}, \frac{4}{5} > \\ &= -\frac{3}{5} + \frac{8}{5} \\ &= 1 \end{aligned}$$

2. This question is looking for the largest possible positive change in $f(x, y)$ at $(2, 0)$. The theorem above that the max is $|\nabla f(2, 0)|$ and it occurs in the direction of $\nabla f(2, 0)$.

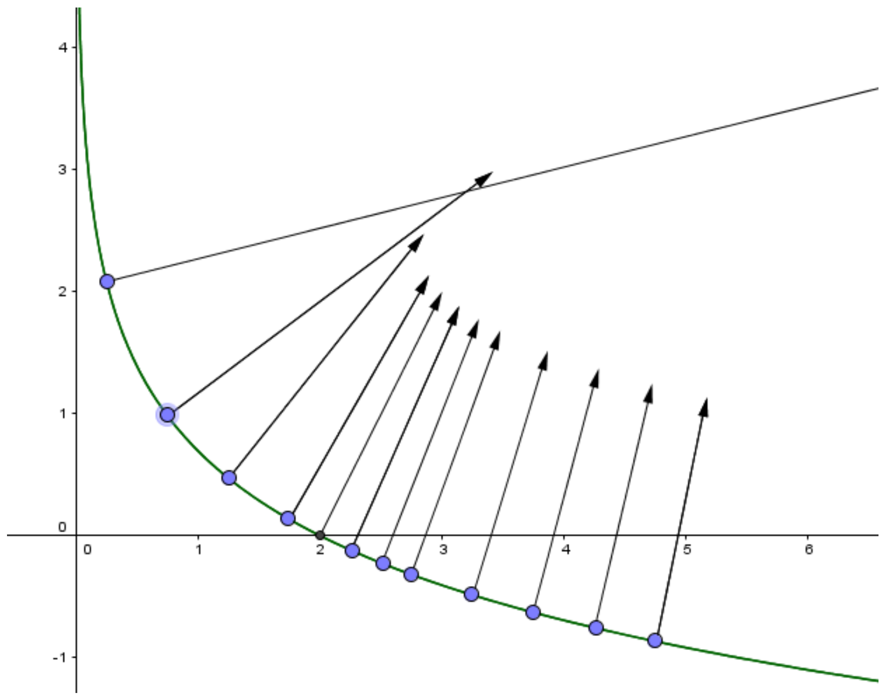
   (a) $\nabla f(2, 0) =< e^0, 2e^0 >=< 1, 2 >$

   (b) The max change is $|\nabla f(2, 0)| = | < 1, 2 > | = \sqrt{1^2 + 2^2} = \sqrt{5}$.



The graph shows both directional vectors. The one on the left points in the direction of $< \frac{-3}{5}, \frac{4}{5} >$ and has a length of 1. The vector on the right points in the direction of $< 2, 1 >$ and has a length of $\sqrt{5}$. It also looks like it points directly up the steepest part of the graph.

The theorem above states that the max positive rate of change occurs in the direction of the gradient. On the graph below I graphed the level curve for $xe^y = 2$. The level curve plots all the points where the plane $z = 2$ intersects $f(x, y) = xe^y$. Note that $(2, 0)$ is on this curve.



I marked other points (x, y) on the graph of $f(x, y) = xe^y$. I specifically choose points (x, y) that would be on the level curve $f(x, y) = 2$. For each point I marked the gradient vector. The important thing is to notice how the gradient vector relates to the level curve.

The gradient vector is always perpendicular to the level curve. This will help explain some concepts later in the chapter.