Math 232

Calculus III

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14.5 The Chain Rule

Definition 1: The Chain Rule, Case 1: One Parameter

Suppose that $z = f(x, y)$ is differentiable in x and y where $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then z is differentiable and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

The formula follows from the previous section of $dz = f_x\, dx + f_y\, dy$.

Example 1

If $z = x^2 y + 3xy^4$, where $x = \sin(2t)$ and $y = \cos(t)$. Find $\dfrac{dz}{dt}$ when $t = 0$.

1. $\dfrac{\partial z}{\partial x} = 2xy + 3y^4$
2. $\dfrac{\partial z}{\partial y} = x^2 + 12xy^3$
3. $\dfrac{dx}{dt} = 2\cos(2t)$
4. $\dfrac{dy}{dt} = -\sin(t)$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy + 3y^4)(2\cos(2t)) + (x^2 + 12xy^3)(-\sin(t))$$

5. Note: We need to find x and y when $t = 0$

$$x = \sin(0) = 0$$

$$y = \cos(0) = 1$$

6. Final:

$$\begin{aligned}\left.\frac{dz}{dt}\right|_{t=0} &(2(0)(1) + 3(1)^4)(2\cos 0) + (0^2 + 12(0)(1)^3)(-\sin 0) \\ &= (0+3)\cdot 2 + (0+0)\cdot 0 \\ &= 6\end{aligned}$$

Definition 2: The Chain Rule, Case 2: Two Parameters

Suppose $z = f(x, y)$ is differentiable and $x = g(s, t)$ and $y = h(s, t)$ are differentiable.

Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Example 2

If $z = e^x \sin(y)$, $x = st^2$, and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

1. Let's find $\frac{\partial z}{\partial s}$
   (a) $\frac{\partial z}{\partial x} = e^x \sin(y)$
   (b) $\frac{\partial z}{\partial y} = e^x \cos(y)$
   (c) $\frac{\partial x}{\partial s} = t^2$
   (d) $\frac{\partial y}{\partial s} = 2st$

$$\frac{\partial z}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

2. Now onto $\frac{\partial z}{\partial t}$
   (a) $\frac{\partial z}{\partial x} = e^x \sin(y)$
   (b) $\frac{\partial z}{\partial y} = e^x \cos(y)$

(c) $\dfrac{\partial x}{\partial s} = 2st$

(d) $\dfrac{\partial y}{\partial s} = s^2$

$$\frac{\partial z}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

Definition 3: General Chain Rule

Suppose z is a differentiable function of x_1, x_2, x_3, ..., x_n with each x_j being a differentiable function of t_1, t_2, t_3, ..., t_m. Then

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + ... + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

Example 3

If $u = x^4y + y^2z^3$, $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2s\sin t$. Find $\dfrac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.

1. Let's with finding $\dfrac{\partial u}{\partial s}$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$$

(a) $\dfrac{\partial u}{\partial x} = 4x^3y$

(b) $\dfrac{\partial u}{\partial y} = x^4 + 2yz^3$

(c) $\dfrac{\partial u}{\partial z} = 3y^2z^2$

(d) $\dfrac{\partial x}{\partial s} = re^t$

(e) $\dfrac{\partial y}{\partial s} = 2rse^{-t}$

(f) $\dfrac{\partial z}{\partial s} = r^2\sin(t)$

$$\frac{\partial u}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-1t}) + (3y^2z^2)(r^2\sin t)$$

2. We are asked to plug in $r = 2$, $s = 1$, and $t = 0$ but $\dfrac{\partial u}{\partial s}$ has x, y and z as well.

(a) $x = (2)(1)e^0 = 2$

(b) $y = (2)(1)^2e^0 = 2$

(c) $z = 2^2(1)\sin(0) = 0$

$$\begin{aligned}\frac{\partial u}{\partial s}_{r=2,s=1,t=0} &= \left[4(2)^3(2)\right]\left[2e^0\right] + \left[2^4 + 2(2)(0)^3\right]\left[2(2)(1)e^0\right] + \left[3(2)^2(0)^2\right]\left[2^2\sin(0)\right] \\ &= 128 + 64 + 0 \\ &= 192\end{aligned}$$

Definition 4: Implicit Differentiation, Case 1

Suppose $F(x, y) = 0$ where $y = f(x)$.

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

Because $\frac{dx}{dx} = 1$ we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{dy}{dx} = 0$$

Solving for $\frac{y}{dx}$ we get

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

where $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$

Definition 5: Implicit Differentiation, Case 2

Suppose $z = f(x, y)$ is defined by $F(x, y, z) = 0$. Then

$$\frac{dz}{dx} = \frac{-F_x}{F_z}$$

$$\frac{dz}{dy} = \frac{-F_y}{F_z}$$

Example 4

Find $\frac{dy}{dx}$ when $x^3 + y^3 = 6xy$ using the techniques developed in calc 1 and from this section.

1. Calc 1: Solve for $\frac{dy}{dx}$

$$\frac{d}{dx}\left[x^3+y^3\right] = \frac{d}{dx}\left[6xy\right]$$
$$3x^2+3y^2\frac{dy}{dx} = 6x\frac{dy}{dx}+6y$$
$$3y^2\frac{dy}{dx}-6x\frac{dy}{dx} = 6y-3x^2$$
$$\frac{dy}{dx}\left[3y^2-6x\right] = 6y-3y^2$$
$$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}$$

2. Calc 3: Rewrite equation as $x^3+y^3-6xy=0$

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

$$\frac{dy}{dx} = \frac{-(3x^2-6y)}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}$$

Example 5

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$.

1. Let $F(x,y,z)=x^3+y^3+z^3+6xyz-1$
2. $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2+6xy)}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$
3. $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$