14.5 The Chain Rule

**Definition 1: The Chain Rule, Case 1: One Parameter**

Suppose that $z = f(x, y)$ is differentiable in $x$ and $y$ where $x = g(t)$ and $y = h(t)$ are differentiable functions of $t$. Then $z$ is differentiable and

$$
\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}
$$

The formula follows from the previous section of $dz = f_x \, dx + f_y \, dy$.

**Example 1**

If $z = x^2y + 3xy^4$, where $x = \sin(2t)$ and $y = \cos(t)$. Find $\frac{dz}{dt}$ when $t = 0$.

1. $\frac{\partial z}{\partial x} = 2xy + 3y^4$
2. $\frac{\partial z}{\partial y} = x^2 + 12xy^3$
3. $\frac{dx}{dt} = 2 \cos(2t)$
4. $\frac{dy}{dt} = - \sin(t)$.

$$
\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}
$$

$$
\frac{dz}{dt} = (2xy + 3y^4)(2 \cos(2t)) + (x^2 + 12xy^3)(- \sin(t))
$$

5. Note: We need to find $x$ and $y$ when $t = 0$

$$
x = \sin(0) = 0
$$

$$
y = \cos(0) = 1
$$
6. Final:
\[
\left. \frac{dz}{dt} \right|_{t=0} = \frac{d}{dt} \left[ 2(0)(1) + 3(1)^4(2 \cos 0) + (0^2 + 12(0)(1)^3)(-\sin 0) \right]
\]
\[
= (0 + 3) \cdot 2 + (0 + 0) \cdot 0
\]
\[
= 6
\]

**Definition 2: The Chain Rule, Case 2: Two Parameters**

Suppose \( z = f(x, y) \) is differentiable and \( x = g(s, t) \) and \( y = h(s, t) \) are differentiable. Then

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}
\]

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}
\]

**Example 2**

If \( z = e^x \sin(y) \), \( x = st^2 \), and \( y = s^2 t \), find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \).

1. Let’s find \( \frac{\partial z}{\partial s} \)
   
   (a) \( \frac{\partial z}{\partial x} = e^x \sin(y) \)
   
   (b) \( \frac{\partial z}{\partial y} = e^x \cos(y) \)
   
   (c) \( \frac{\partial x}{\partial s} = t^2 \)
   
   (d) \( \frac{\partial y}{\partial s} = 2st \)

   \[
   \frac{\partial z}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)
   \]

2. Now onto \( \frac{\partial z}{\partial t} \)
   
   (a) \( \frac{\partial z}{\partial x} = e^x \sin(y) \)
   
   (b) \( \frac{\partial z}{\partial y} = e^x \cos(y) \)
(c) $\frac{\partial x}{\partial s} = 2st$

(d) $\frac{\partial y}{\partial s} = s^2$

$$\frac{\partial z}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

**Definition 3: General Chain Rule**

Suppose $z$ is a differentiable function of $x_1, x_2, x_3, ..., x_n$ with each $x_j$ being a differentiable function of $t_1, t_2, t_3, ..., t_m$. Then

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + ... + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

**Example 3**

If $u = x^4y + y^2z^3$, $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2s\sin t$. Find $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.

1. Let’s with finding $\frac{\partial u}{\partial s}$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$$

(a) $\frac{\partial u}{\partial x} = 4x^3y$

(b) $\frac{\partial u}{\partial y} = x^4 + 2yz^3$

(c) $\frac{\partial u}{\partial z} = 3y^2z^2$

(d) $\frac{\partial x}{\partial s} = re^t$

(e) $\frac{\partial y}{\partial s} = 2rse^{-t}$

(f) $\frac{\partial z}{\partial s} = r^2\sin(t)$

$$\frac{\partial u}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-1t}) + (3y^2z^2)(r^2\sin t)$$

2. We are asked to plug in $r = 2$, $s = 1$, and $t = 0$ but $\frac{\partial u}{\partial s}$ has $x$, $y$ and $z$ as well.
(a) \( x = (2)(1)e^0 = 2 \)

(b) \( y = (2)(1)^2e^0 = 2 \)

(c) \( z = 2^2(1)\sin(0) = 0 \)

\[
\frac{\partial u}{\partial s_{r=2,s=1,t=0}} = [4(2)^3(2)] [2e^0] + [2^4 + 2(2)(0)^3] [2(2)(1)e^0] + [3(2)^2(0)^2] [2^2 \sin(0)]
\]

\[= 128 + 64 + 0 \]

\[= 192 \]

**Definition 4: Implicit Differentiation, Case 1**

Suppose \( F(x, y) = 0 \) where \( y = f(x) \).

\[
\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0
\]

Because \( \frac{dx}{dx} = 1 \) we get

\[
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0
\]

Solving for \( \frac{dy}{dx} \) we get

\[
\frac{dy}{dx} = -\frac{F_x}{F_y}
\]

where \( F_x = \frac{\partial F}{\partial x} \) and \( F_y = \frac{\partial F}{\partial y} \)

**Definition 5: Implicit Differentiation, Case 2**

Suppose \( z = f(x, y) \) is defined by \( F(x, y, z) = 0 \). Then

\[
\frac{dz}{dx} = -\frac{F_x}{F_z}
\]

\[
\frac{dz}{dy} = -\frac{F_y}{F_z}
\]

**Example 4**

Find \( \frac{dy}{dx} \) when \( x^3 + y^3 = 6xy \) using the techniques developed in calc 1 and from this section.
1. Calc 1: Solve for $\frac{dy}{dx}$

\[
\frac{d}{dx} [x^3 + y^3] = \frac{d}{dx} [6xy]
\]

\[
3x^2 + 3y^2 \frac{dy}{dx} = 6x \frac{dy}{dx} + 6y
\]

\[
3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2
\]

\[
\frac{dy}{dx} [3y^2 - 6x] = 6y - 3y^2
\]

\[
\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}
\]

2. Calc 3: Rewrite equation as $x^3 + y^3 - 6xy = 0$

\[
\frac{dy}{dx} = \frac{-F_x}{F_y}
\]

\[
\frac{dy}{dx} = \frac{-3x^2 - 6y}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}
\]

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**Example 5**

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

1. Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$

2. $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-3x^2 + 6xy}{3z^2 + 6xy} = \frac{x^2 + 2yz}{z^2 + 2xy}$

3. $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{3y^2 + 6xz}{3z^2 + 6xy} = \frac{y^2 + 2xz}{z^2 + 2xy}$