Math 232

Calculus III

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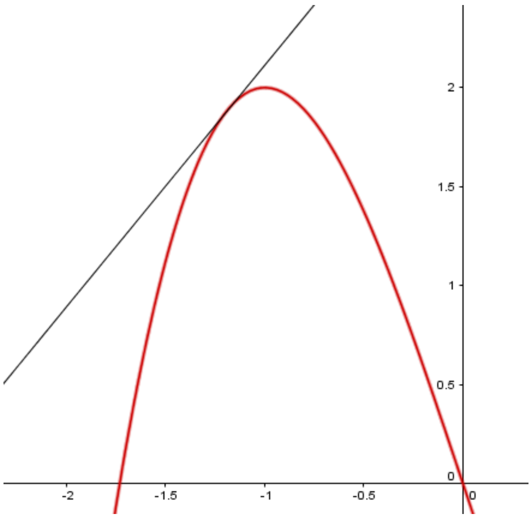
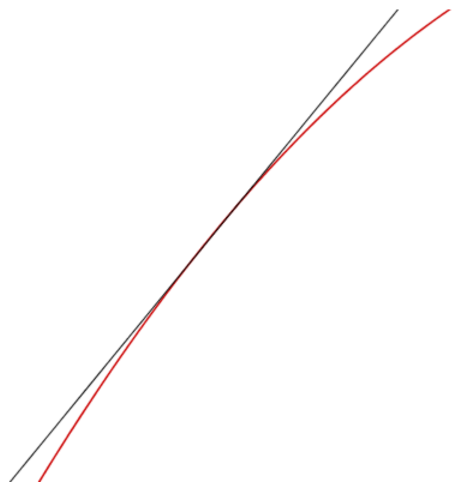
14.4 Tangent Planes and Linear Approximations

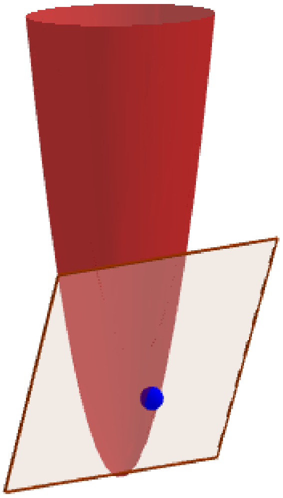
Zoom in on a single variable function and it looks like a straight line. We call this the tangent line at a point (x_0, y_0).





The graph on the right is what it looks like after zooming in a bit.

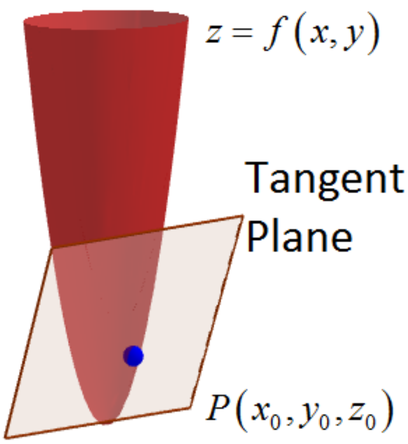
In two variables we don't have tangent lines. Since functions of two variables are surfaces, when we zoom in at a given point (x_0, y_0, z_0) it will look like a plane. In this section we will discuss tangent planes, how to find them, and what we can do with them.



The left graph shows a curve and what looks like a line going through a point. I rotated the graph slightly and you can see it actually is a plane that goes through the point. If you zoom in on that point you won't be able to tell the difference between the curve and the tangent line.

Definition 1: Equation of a Tangent Plane

Suppose a surface S has the equation $z = f(x, y)$ such that f_x and f_y are continuous and let $P(x_0, y_0, z_0)$ be a point on S. Then the equation for the tangent plane to the surface $z = f(x, y)$ at P is



$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1

Find the tangent plane on $z = x^2 + x\cos(y)$ at the point $P(\pi/, \pi/2)$.

1. Find f_x and $f_x(\pi, \pi/2)$

$$f_x = 2x + \cos(y)$$

$$f_x(\pi, \pi/2) = 2\pi + \cos(\pi/2) = 2\pi$$

2. Find f_y and $f_y(\pi, \pi/2)$

$$f_y = -x\sin(y)$$

$$f_y(\pi, \pi/2) = -\pi\sin(\pi/2) = -\pi$$

3. Find z_0

$$z_0 = (\pi)^2 + \pi(\cos(\pi/2)) = \pi^2$$

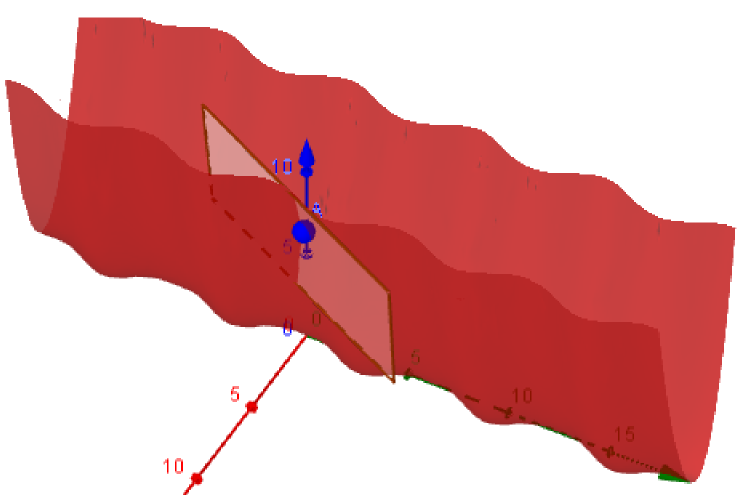
4. Use the formula for the tangent plane

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - \pi^2 = 2\pi(x - \pi) - \pi(y - \pi/2)$$

$$z = 2\pi x - 2\pi^2 - \pi y + \pi^2/2 + \pi^2$$

$$z = 2\pi x - \pi y - \pi^2/2$$



Example 2

Find the tangent plane on $z = 2x^2 + y^2$ at the point $P(1, 1, 3)$.

1. Find f_x and $f_x(1, 1)$

$$f_x = 4x$$

$$f_x(1, 1,) = 4$$

2. Find f_y and $f_y(1, 1)$

$$f_y = 2y$$

$$f_y(1, 1) = 2$$

3. $z_0 = 3$

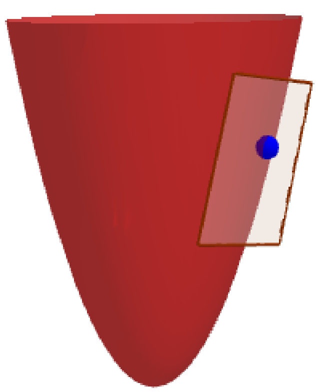
4. Use the formula for the tangent plane

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 3 = 4(x - 1) + 2(y - 1)$$

$$z - 3 = 4x - 4 + 2y - 2$$

$$z = 4x + 2y - 3$$



Definition 2: Linear Approximation

The tangent plane $L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$ is also called the linear approximation. We can use to to approximate z values near $P(x_0, y_0)$.

Example 3

For the previous problem the linear approximation could be written as

$$L(x, y) = 4x + 2y - 3$$

Suppose we want to estimate $f(1.1, 0.95)$.

1. Actual Value: $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$

2. Linear Approximation:

$$f(1.1, 0.95) \approx L(1.1, 0.95) = 4(1.1) + 2(0.95) - 3 = 3.3$$

Keep in mind that the approximation gets worse as you move away from the point $(1, 1, 3)$.

1. $f(2, 3) = 2(2)^2 + 3^2 = 17$

2. $L(2, 3) = 4(2) + 2(3) - 3 = 11$

Example 4

Show $xe^{xy} \approx x + y$ near $P(1, 0)$.

Since $x + y$ is a plane in $\mathbb{R}$, it's really asking us to verify if $z = x + y$ is the tangent plane at $P(1, 0)$.

1. Let $f(x, y) = xe^{xy}$

2. Find f_x and $f_x(1, 0)$

$$f_x = 1e^{xy} + xe^{xy} \cdot y = e^{xy} + xye^{xy}$$

$$f_x(1, 0) = e^0 + 1(0)e^0 = 1$$

3. Find f_y and $f_y(1, 0)$

$$f_y = xe^{xy} \cdot x = x^2e^{xy}$$

$$f_y(1, 0) = 1^2e^0 = 1$$

4. $z_0 = 1e^0 = 1$

5. Use the formula for the linear approximation (tangent plane)

$$z - z_0 = f_x(1,0)(x - x_0) + f_y(1,0)(y - y_0)$$

$$z - 1 = 1(x - 1) + 1(y - 0)$$

$$z = x + y$$