14.4 Tangent Planes and Linear Approximations

Zoom in on a single variable function and it looks like a straight line. We call this the tangent line at a point \((x_0, y_0)\).

The graph on the right is what it looks like after zooming in a bit.

In two variables we don’t have tangent lines. Since functions of two variables are surfaces, when we zoom in at a given point \((x_0, y_0, z_0)\) it will look like a plane. In this section we will discuss tangent planes, how to find them, and what we can do with them.
The left graph shows a curve and what looks like a line going through a point. I rotated the graph slightly and you can see it actually is a plane that goes through the point. If you zoom in on that point you won’t be able to tell the difference between the curve and the tangent line.

**Definition 1: Equation of a Tangent Plane**

Suppose a surface $S$ has the equation $z = f(x, y)$ such that $f_x$ and $f_y$ are continuous and let $P(x_0, y_0, z_0)$ be a point on $S$. Then the equation for the tangent plane to the surface $z = f(x, y)$ at $P$ is

$$z - z_0 = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$$
Example 1

Find the tangent plane on \( z = x^2 + x \cos(y) \) at the point \( P(\pi/, \pi/2) \).

1. Find \( f_x \) and \( f_x(\pi, \pi/2) \)

\[
\begin{align*}
    f_x &= 2x + \cos(y) \\
    f_x(\pi, \pi/2) &= 2\pi + \cos(\pi/2) = 2\pi
\end{align*}
\]

2. Find \( f_y \) and \( f_y(\pi, \pi/2) \)

\[
\begin{align*}
    f_y &= -x \sin(y) \\
    f_y(\pi, \pi/2) &= -\pi \sin(\pi/2) = -\pi
\end{align*}
\]

3. Find \( z_0 \)

\[
    z_0 = (\pi)^2 + \pi(\cos(\pi/2)) = \pi^2
\]

4. Use the formula for the tangent plane

\[
    z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
\]

\[
    z - \pi^2 = 2\pi(x - \pi) - \pi(y - \pi/2)
\]

\[
    z = 2\pi x - 2\pi^2 - \pi y + \pi^2/2 + \pi^2
\]

\[
    z = 2\pi x - \pi y - \pi^2/2
\]
Example 2

Find the tangent plane on \( z = 2x^2 + y^2 \) at the point \( P(1, 1, 3) \).

1. Find \( f_x \) and \( f_x(1, 1) \)
   \[
   f_x = 4x \\
   f_x(1, 1) = 4
   \]

2. Find \( f_y \) and \( f_y(1, 1) \)
   \[
   f_y = 2y \\
   f_y(1, 1) = 2
   \]

3. \( z_0 = 3 \)

4. Use the formula for the tangent plane
   \[
   z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
   \]
   \[
   z - 3 = 4(x - 1) + 2(y - 1) \\
   z - 3 = 4x - 4 + 2y - 2 \\
   z = 4x + 2y - 3
   \]

Definition 2: Linear Approximation

The tangent plane \( L(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0 \) is also called the linear approximation. We can use it to approximate \( z \) values near \( P(x_0, y_0) \).
Example 3

For the previous problem the linear approximation could be written as

\[ L(x, y) = 4x + 2y - 3 \]

Suppose we want to estimate \( f(1.1, 0.95) \).

1. Actual Value: \( f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225 \)

2. Linear Approximation:

\[ f(1.1, 0.95) \approx L(1.1, 0.95) = 4(1.1) + 2(0.95) - 3 = 3.3 \]

Keep in mind that the approximation gets worse as you move away from the point \((1, 1, 3)\).

1. \( f(2, 3) = 2(2)^2 + 3^2 = 17 \)

2. \( L(2, 3) = 4(2) + 2(3) - 3 = 11 \)

Example 4

Show \( xe^{xy} \approx x + y \) near \( P(1,0) \).

Since \( x + y \) is a plane in \( \mathbb{R} \), it’s really asking us to verify if \( z = x + y \) is the tangent plane at \( P(1,0) \).

1. Let \( f(x, y) = xe^{xy} \)

2. Find \( f_x \) and \( f_x(1,0) \)

\[ f_x = 1e^{xy} + x e^{xy} \cdot y = e^{xy} + xy e^{xy} \]

\[ f_x(1,0) = e^0 + 1(0)e^0 = 1 \]

3. Find \( f_y \) and \( f_y(1,0) \)

\[ f_y = xe^{xy} \cdot x = x^2e^{xy} \]

\[ f_y(1,0) = 1^2e^0 = 1 \]
4. \( z_0 = 1e^0 = 1 \)

5. Use the formula for the linear approximation (tangent plane)

\[
z - z_0 = f_x(1, 0)(x - x_0) + f_y(1, 0)(y - y_0) \\
z - 1 = 1(x - 1) + 1(y - 0) \\
z = x + y
\]