Math 232

Calculus III

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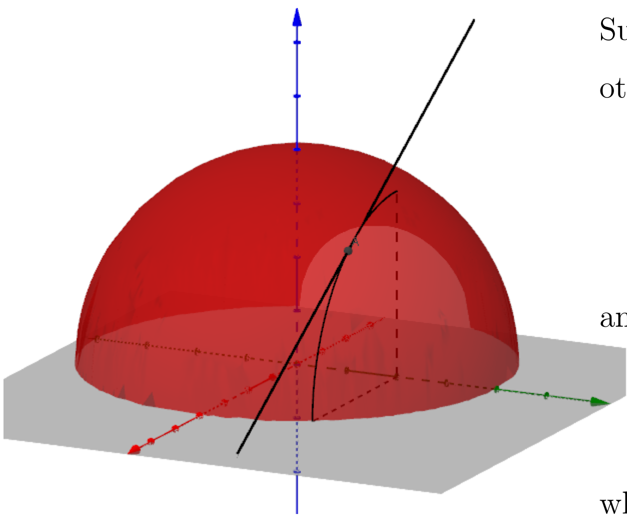
14.3 Partial Derivatives

Definition 1: Partial Derivatives

Let f be a function of two variables.



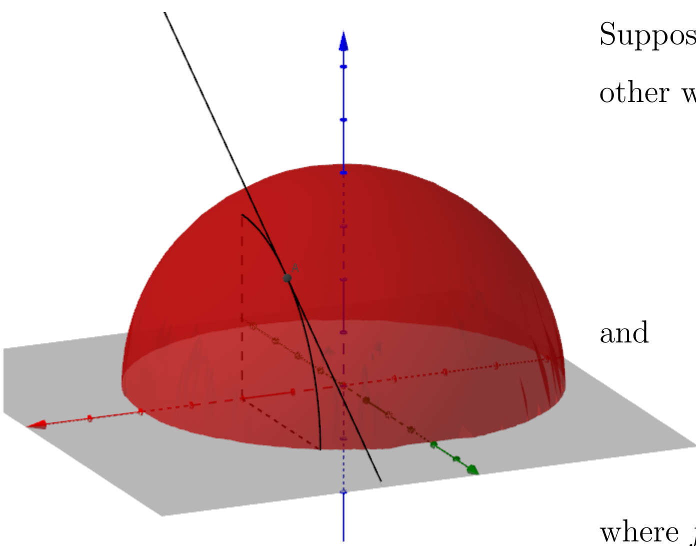
Suppose we let y be fixed and vary x. In other words

$$g(x) = f(x, y)$$

and

$$g'(x) = f_x(x, y)$$

where $f_x(x, y)$ is called the Partial Derivative of f with respect to x.



Suppose we let x be fixed and vary y. In other words

$$h(y) = f(x, y)$$

and

$$h'(y) = f_y(x, y)$$

where $f_y(x, y)$ is called the Partial Derivative of f with respect to y.

The partial derivatives are defined in a similar way to how we defined derivatives of one variable (with limits).

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Luckily just like derivative in one variable we do have all the normal rules: product, quotient, chain, etc. Before we get into some examples let's discuss the different notation used.

Definition 2: Notation for Partial Derivatives

Let $z = f(x,y)$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = D_y f$$

In order to find f_x you must regard y as a constant and differentiate with respect to x. To find f_y you must regard x as a constant and differentiate with respect to y

Example 1

Let $f(x,y) = x^3 + 2x^2y^3 - 4y^2 + 6$. Find f_x, f_y, $f_x(1,2)$, and $f_y(1,2)$.

$$f_x = 3x^2 + 4xy^3 \qquad f_x(1,2) = 3(1)^2 + 4(1)(2)^3 = 35$$

$$f_y = 6x^2y^2 - 8y \qquad f_y(1,2) = 6(1)^2(2)^2 - 8(2) = 8$$

Example 2

Let $f(x,y) = \sin\left(\dfrac{x}{1+y}\right)$. Find f_x and f_y.

$$\begin{aligned} f_x &= \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left[\frac{x}{1+y}\right] \\ &= \cos\left(\frac{x}{1+y}\right) \cdot \left(\frac{1}{1+y}\right) \end{aligned}$$

$$\begin{aligned} f_y &= \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left[\frac{x}{1+y}\right] \\ &= \cos\left(\frac{x}{1+y}\right) \cdot \left(\frac{-x}{(1+y)^2}\right) \end{aligned}$$

Partial derivatives work the same for any number of variables. The next example involves a function of three variables.

Example 3

Let $f(x, y, z) = e^{xyz} \ln(z)$. Find f_x, f_y, and f_z.

$$f_x = yze^{xyz} \ln(z)$$

$$f_y = xze^{xyz} \ln(z)$$

$$f_z = xye^{xyz} \ln(z) + e^{xyz} \frac{1}{z} = e^{xyz} \left(xy \ln(z) + \frac{1}{z} \right)$$

In calculus I you learned about implicity differentiation. This involves treating all variables as a function of one single variable. For example x could be the independent variable and

$$z = f(x)$$

$$y = f(x)$$

Example 4: Simple Examples

$$\frac{\partial}{\partial x} [z] = \frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x} \left[3z^4\right] = 12z^3 \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial x} \left[e^{zy}\right] = ye^{zy} \cdot \frac{\partial z}{\partial x}$$

Example 5

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^3 + y^3 + z^3 + 6xyz = 1$

$$\begin{aligned}
\frac{\partial}{\partial x}\left[x^3+y^3+z^3+6xyz\right] &= \frac{\partial}{\partial x}[1] \\
3x^2+3z^2\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x} &= 0 \\
\frac{\partial z}{\partial x}\left[3z^2+6xy\right] = -3x^2-6yz & \\
\frac{\partial z}{\partial x} = \frac{-3x^2-6yz}{3z^2+6xy} &
\end{aligned}$$

Following the same idea:

$$\frac{\partial z}{\partial y} = \frac{-3y^2-6xz}{3z^2+6xy}$$

Definition 3: Higher Order Partial Derivatives

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

f_{xy} and f_{yx} are called mixed partial derivatives.

Example 6

Find all second partial derivatives for $f(x,y) = x^3 + x^2y^3 - 2y^2$

1. First start by finding f_x

$$f_x = 3x^2 + 2xy^3$$

   (a) Find f_{xx}

$$f_{xx} = 6x + 2y^3$$

(b) Find f_{xy}

$$f_{xy} = 6xy^2$$

2. Now find f_y

$$f_y = 3x^2y^2 - 4y$$

(a) Find f_{yy}

$$f_{yy} = 6x^2y - 4$$

(b) Find f_{yx}

$$f_{yx} = 6xy^2$$

It's actually no coincidence that $f_{xy} = f_{yx}$. We have a theorem that says so.

Theorem 1: Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If f_{xy} and f_{yx} are both continuous, then

$$f_{xy} = f_{yx}$$

In fact changing the order of partial differentiation will not matter. For example,

$$f_{xxy} = f_{xyx} = f_{yxx}$$