14.2 Limits and Continuity

In this section our goal is to evaluate limits of the form

\[ \lim_{(x,y) \to (a,b)} f(x, y) = L \]

Let’s take a look back at limits in one variable.

There are only two ways to approach the point \( x = a \). That’s why we tried to show a limit exists we only looked at the left and right hand limit. This is because the direction we approach \( y \) is determined by \( x \). So what happens if \( x \) and \( y \) are both independent?

When the variables \( x \) and \( y \) are independent you can approach the point \((a, b)\) from infinity many ways. Keep in mind that this limit is suppose to be extended to the three-dimensional space. You can approach a point from above, below, right, left, diagonally, etc.

Note: If \( f(x, y) \to L_1 \) as \((x, y) \to (a, b)\) along Path \( C_1 \) and \( f(x, y) \to L_2 \) as \((x, y) \to (a, b)\)
along Path $C_2$ then $\lim_{(x,y) \to (a,b)} f(x,y)$ does not exist.

To show a limit exists, we must show $\lim_{(x,y) \to (a,b)} f(x,y) = L$ along every Path $C_i$.

**Theorem 1**

If $f(x, y)$ is continuous at $(a, b)$, then $\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$.

**Example 1**

Evaluate $\lim_{(x,y) \to (0,1)} \frac{x^2 - y^2}{x^2 + y^2}$

Since $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$ is continuous for all $(x, y) \in \mathbb{R}^2$ except $(0,0)$, we can just plug in $(0,1)$.

$$\lim_{(x,y) \to (0,0)} \frac{0^2 - 1^2}{0^2 + 1^2} = -1$$

Notice that whatever direction you to take approach $(x,y) = (0,1)$ it leads to the point $(0,1,-1)$.

**Example 2**

Evaluate $\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$.

In this example $(0,0)$ is the only point where $f(x,y)$ is discontinuous. But that doesn’t prove a limit does not exist.

We must consider every path $C$ where $(x, y) \to (0,0)$. Consider the graph of $f(x,y)$. 
Notice that when you approach along the $x$-axis it appears the $z$-value will be 1.

When you approach along the $y$-axis it appears the $z$-value is -1. This shows the limit will not exist. Let’s prove it.

Let’s consider two specific paths.

1. Along the $x$-axis. This means we’re looking at all points of the form $(x, 0)$.

$$\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{x \to 0} f(x, 0)$$

$$= \lim_{x \to 0} \frac{x^2 - 0^2}{x^2 + 0^2} = 1$$

2. Along the $y$-axis. This means we’re looking at all points of the form $(0, y)$.

$$\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{y \to 0} f(0, x)$$

$$= \lim_{y \to 0} \frac{0^2 - y^2}{0^2 + y^2} = -1$$

3. Since the limit equals 1 along path $C_1$ ($x$-axis) and -1 along path $C_2$ ($y$-axis), the limit does not exist.

**Example 3**

Evaluate $\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}$.

Since $f(x, y)$ is not continuous at $(0, 0)$ we must consider different paths to approach $(0, 0)$. 

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1. Along the $x$-axis. This means we’re looking at all points of the form $(x, 0)$.

\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{x \to 0} f(x, 0) = \lim_{x \to 0} \frac{x(0)}{x^2 + 0^2} = 0
\]

2. Along the $y$-axis. This means we’re looking at all points of the form $(0, y)$.

\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{y \to 0} f(0, y) = \lim_{y \to 0} \frac{x(0)}{0^2 + y^2} = 0
\]

3. Notice that the limit is $L = 0$ for both directions. But this is only two of infinitely many approaches. Showing a limit exists is much harder than showing one doesn’t exist.

4. Along the path $y = x$. This means we’re looking at all points of the form $(x, x)$.

\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{x \to 0} f(x, x) = \lim_{x \to 0} \frac{x \cdot x}{x^2 + x^2} = \lim_{x \to 0} \frac{x^2}{2x^2} = \frac{1}{2}
\]

5. The limit does not exist. We found two paths where $f(x, y)$ approaches two different limit values $L_1 = 1/2$ and $L_2 = 0$. The graph of $f(x, y) = \frac{xy}{x^2 + y^2}$ supports this.
Example 4

Evaluate \( \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} \)

1. Along the \( x \)-axis.
\[
\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{x \to 0} \frac{x(0)^2}{x^2 + 0^4} = 0
\]

2. Along the \( y \)-axis.
\[
\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{y \to 0} \frac{0y^2}{0^2 + y^4} = 0
\]

3. Along \( y = x \)
\[
\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{x \to 0} \frac{x \cdot x^2}{x^2 + x^4} = \lim_{x \to 0} \frac{x^3}{x^2 + x^4} = \lim_{x \to 0} \frac{x^3}{1 + x^2} = 0
\]

4. So far we get getting zero. But all these approaches are linear. Why not try a quadratic approach, for example \( y = x^2 \). This means we’re looking at all points of the form \((x, x^2)\).
\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{x \to 0} f(x, x^2) = \lim_{x \to 0} \frac{x(x^2)^2}{x^2 + x^8} = \lim_{x \to 0} \frac{x^5}{x^2 + x^8} = 0
\]

5. Dang. How about \( x = y^2 \)
\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{y \to 0} f(y^2, y) = \lim_{y \to 0} \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \lim_{y \to 0} \frac{y^4}{2y^4} = \frac{1}{2}
\]

6. \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist. Check out the graph of \( f(x, y) \).
So how in the world does one show a limit does exist? Normally it involves luck and the squeeze theorem.

**Theorem 2: Squeeze Theorem**

Let $f(x, y) \leq g(x, y) \leq h(x, y)$ in a disk around $(a, b)$ and $\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(a,b)} h(x, y) = L$. Then

$$\lim_{(x,y)\to(a,b)} g(x, y) = L$$

**Example 5: Uses the Squeeze Theorem**

Evaluate $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}$. When you go along the $x$-axis, $y$-axis, $y = x$, $y = x^2$, etc., you keep getting $L = 0$. After awhile you assume the limit does exist and it’s $L = 0$.

We’re going to use an application of the Squeeze Theorem

**Theorem 3: Application of Squeeze Theorem**

If $0 \leq |g(x, y)| \leq h(x, y)$ in a disk around $(a, b)$ and $\lim_{(x,y)\to(a,b)} h(x, y) = 0$, then

$$\lim_{(x,y)\to(a,b)} g(x, y) = 0$$
Let’s consider \(|3x^2y|\). If we want to find a bigger function \(h(x, y)\) we can try making the numerator bigger.

\[x^2y < (x^2 + y^2)y\] because \(x^2 < x^2 + y^2\)

It follows that

\[0 \leq \left| \frac{3x^2y}{x^2 + y^2} \right| < \frac{|(x^2 + y^2)y|}{x^2 + y^2} = 3|y|\]

Since \(\lim_{(x,y)\to(0,0)} 3|y| = 0\) we satisfy the above theorem.

\[\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0 \text{ (IT EXISTS!)}\]

The last type of problem involves writing \(f(x, y)\) into a polar function using

\[x = r \cos(\theta), \quad y = r \sin(\theta), \quad x^2 + y^2 = r^2\]

**Example 6: Converting to Polar**

Let’s try evaluating \(\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}\) again.

If \((x, y) \to (0, 0)\) is the same thing as saying \(r \to 0\). If \(r \to 0\) it really doesn’t matter what \(\theta\) is.

\[
\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = \lim_{r \to 0} \frac{3r^2 \cos^2(\theta) \cdot r \sin(\theta)}{r^2} \\
= \lim_{r \to 0} \frac{3r^3 \cos^2(\theta) \sin(\theta)}{r^2} \\
= \lim_{r \to 0} 3r \cos^2(\theta) \sin(\theta) = 0
\]