MATH 232

CALCULUS III

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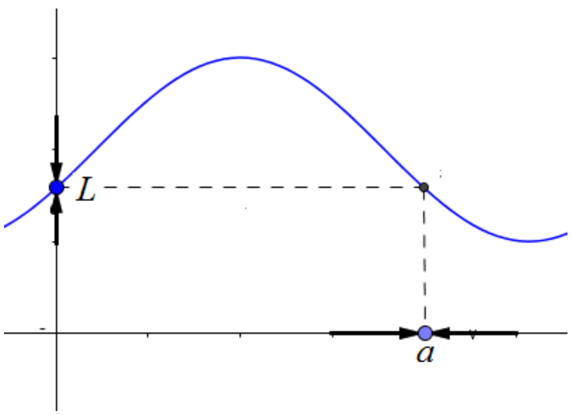
14.2 Limits and Continuity

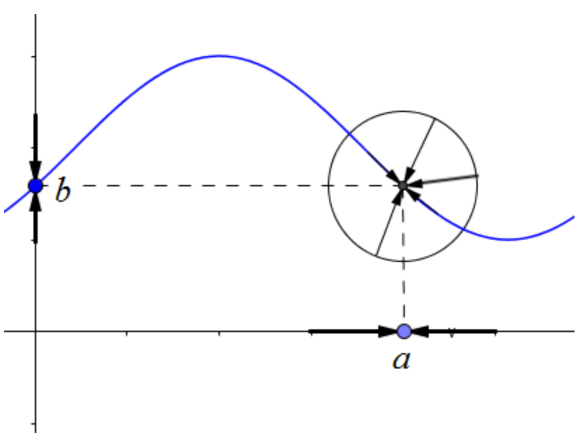
In this section our goal is to evaluate limits of the form

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Let's take a look back at limits in one variable.



There are only two ways to approach the point $x = a$. That's why we tried to show a limit exists we only looked at the left and right hand limit. This is because the direction we approach y is determined by x. So what happens if x and y are both independent?



When the variables x and y are independent you can approach the point (a, b) from infinity many ways. Keep in mind that this limit is suppose to be extended to the three-dimensional space. You can approach a point from above, below, right, left, diagonally, etc.

Note: If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along Path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$

along Path C_2 then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

To show a limit exists, we must show $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$ along every Path C_i.

Theorem 1

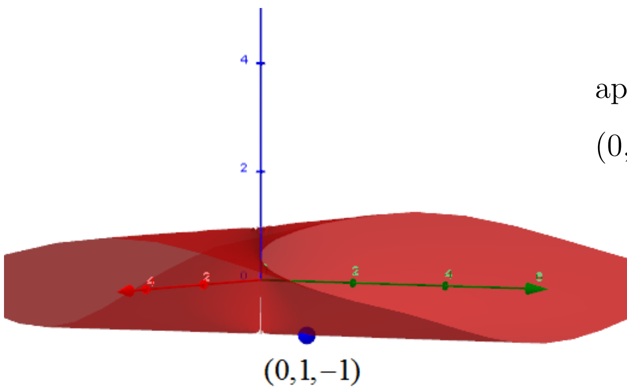
If $f(x,y)$ is continuous at (a,b), then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example 1

Evaluate $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x^2 - y^2}{x^2 + y^2}$

Since $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is continuous for all $(x,y) \in \mathbb{R}^2$ except $(0,0)$, we can just plug in $(0,1)$.

$$\lim_{(x,y)\to(0,0)} \frac{0^2 - 1^2}{0^2 + 1^2} = -1$$



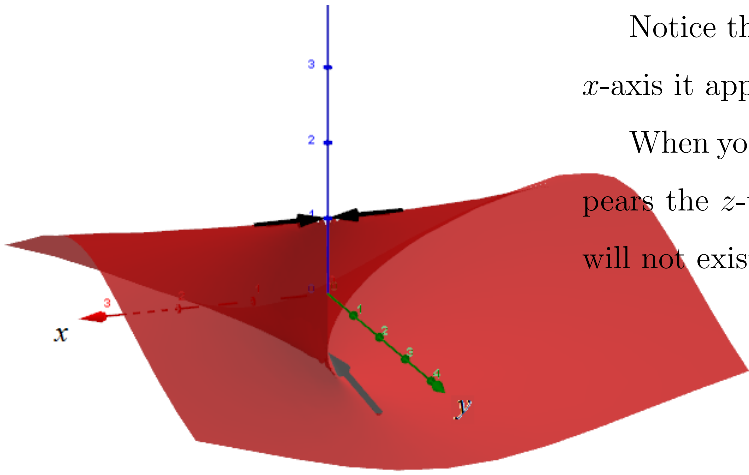
Notice that whatever direction you to take approach $(x,y) = (0,1)$ it leads to the point $(0,1,-1)$.

Example 2

Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$.

In this example $(0,0)$ is the only point where $f(x,y)$ is discontinuous. But that doesn't prove a limit does not exist.

We must consider every path C where $(x,y) \to (0,0)$. Consider the graph of $f(x,y)$.



Notice that when you approach along the x-axis it appears the z-value will be 1.

When you approach along the y-axis it appears the z-value is -1. This shows the limit will not exist. Let's prove it.

Let's consider two specific paths.

1. Along the x-axis. This means we're looking at all points of the form $(x, 0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0)$$

$$= \lim_{x\to 0} \frac{x^2 - 0^2}{x^2 + 0^2} = 1$$

2. Along the y-axis. This means we're looking at all points of the form $(0, y)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,x)$$

$$= \lim_{y\to 0} \frac{0^2 - y^2}{0^2 + y^2} = -1$$

3. Since the limit equals 1 along path C_1 (x-axis) and -1 along path C_2 (y-axis), the limit does not exist.

Example 3

Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$.

Since $f(x, y)$ is not continuous at $(0, 0)$ we must consider different paths to approach $(0, 0)$.

1. Along the x-axis. This means we're looking at all points of the form $(x, 0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0)$$
$$= \lim_{x\to 0} \frac{x(0)}{x^2+0^2} = 0$$

2. Along the y-axis. This means we're looking at all points of the form $(0, y)$.

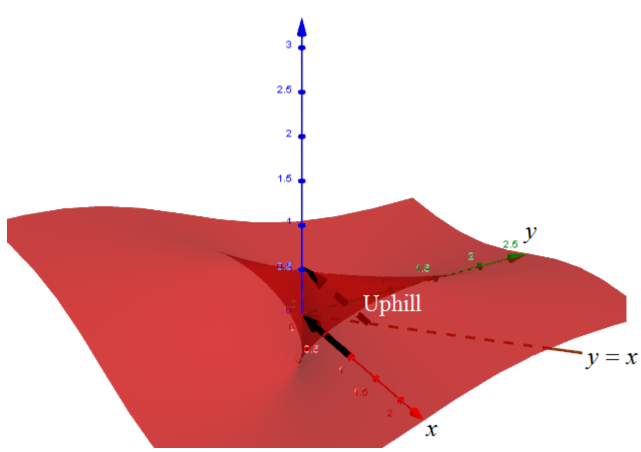
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y)$$
$$= \lim_{y\to 0} \frac{x(0)}{0^2+y^2} = 0$$

3. Notice that the limit is $L = 0$ for both directions. But this is only two of infinitely many approaches. Showing a limit exists is much harder than showing one doesn't exist.

4. Along the path $y = x$. This means we're looking at all points of the form (x, x).

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x)$$
$$= \lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

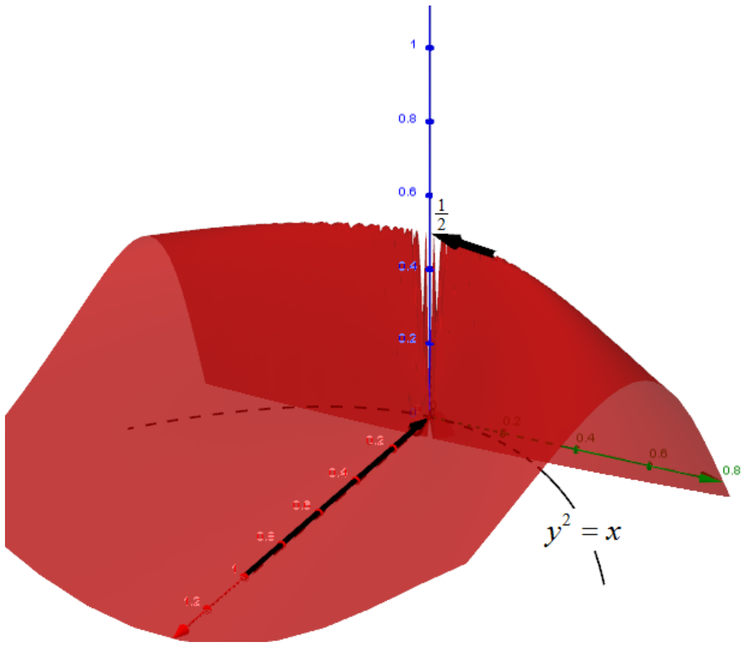
5. The limit does not exist. We found two paths where $f(x, y)$ approaches two different limit values $L_1 = 1/2$ and $L_2 = 0$. The graph of $f(x,y) = \dfrac{xy}{x^2+y^2}$ supports this.



Example 4

Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

1. Along the x-axis.
$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x(0)^2}{x^2+0^4} = 0$$
2. Along the y-axis.
$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{0y^2}{0^2+y^4} = 0$$
3. Along $y = x$
$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim x \to 0 \frac{x \cdot x^2}{x^2+x^4}$$
$$= \lim_{x\to 0} \frac{x^3}{x^2+x^4} = \lim_{x\to 0} \frac{x}{1+x^2} = 0$$
4. So far we get getting zero. But all these approaches are linear. Why not try a quadratic approach, for example $y = x^2$. This means we're looking at all points of the form (x, x^2).
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x^2)$$
$$= \lim_{x\to 0} \frac{x(x^2)^2}{x^2+x^8} = \lim_{x\to 0} \frac{x^5}{x^2+x^8} = 0$$
5. Dang. How about $x = y^2$
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(y^2,y)$$
$$= \lim_{y\to 0} \frac{y^2 \cdot y^2}{(y^2)^2+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$
6. $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Check out the graph of $f(x,y)$.



So how in the world does one show a limit does exist? Normally it involves luck and the squeeze theorem.

Theorem 2: Squeeze Theorem

Let $f(x,y) \leq g(x,y) \leq h(x,y)$ in a disk around (a,b) and $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = L$. Then

$$\lim_{(x,y)\to(a,b)} g(x,y) = L$$

Example 5: Uses the Squeeze Theorem

Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$.

When you go along the x-axis, y-axis, $y = x$, $y = x^2$, etc., you keep getting $L = 0$. After awhile you assume the limit does exist and it's $L = 0$.

We're going to use an application of the Squeeze Theorem

Theorem 3: Application of Squeeze Theorem

If $0 \leq |g(x,y)| \leq h(x,y)$ in a disk around (a,b) and $\lim_{(x,y)\to(a,b)} = 0$, then

$$\lim_{(x,y)\to(a,b)} g(x,y) = 0$$

Let's consider $\left|\dfrac{3x^2y}{x^2+y^2}\right|$. If we want to find a bigger function $h(x,y)$ we can try making the numerator bigger.

$$x^2y < (x^2+y^2)y \text{ because } x^2 < x^2+y^2$$

It follows that

$$0 \leq \left|\frac{3x^2y}{x^2+y^2}\right| < \frac{|(x^2+y^2)y|}{x^2+y^2} = 3|y|$$

Since $\lim_{(x,y)\to(0,0)} 3|y| = 0$ we satisfy the above theorem.

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0 \text{ (IT EXISTS!)}$$

The last type of problem involves writing $f(x,y)$ into a polar function using

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad x^2+y^2 = r^2$$

Example 6: Converting to Polar

Let's try evaluating $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ again.

If $(x,y) \to (0,0)$ is the same thing as saying $r \to 0$. If $r \to 0$ it really doesn't matter what θ is.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} &= \lim_{r\to 0} \frac{3r^2\cos^2(\theta)\cdot r\sin(\theta)}{r^2} \\ &= \lim_{r\to 0} \frac{3r^3\cos^2(\theta)\sin(\theta)}{r^2} \\ &= \lim_{r\to 0} 3r\cos^2(\theta)\sin(\theta) = 0\end{aligned}$$