Math 232

Calculus III

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14.1 Functions of Several Variables

Example 1: Examples of Functions with Several Variables

Consider a temperature function T with variables x =longitude and y =laditude. We can write it as $T(x, y)$.

Temperature can also depend on other variables such as time t and elevation e giving us $T(x, y, t, e)$.

Volume of a cone: $V = \pi r^2 h = f(r, h)$

Most of our problems will be of the form $z = f(x, y)$ where x and y are independent and x is dependent.

Definition 1

A function f of two variables is a rule that assigns to each ordered pair (x, y) in a domain D a real number denoted by $z = f(x, y)$.

Example 2

Find the domain of $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$. Then find $f(3, 2)$.

Finding the domain of $f(x, y)$ is not much different than finding the domain of $f(x)$. You try to find all the values of x and y that can be plugged in. The only difference is that the domain of x may relay on y and vice versa. Let's look at the two parts of $f(x, y)$ that can give us some problems.

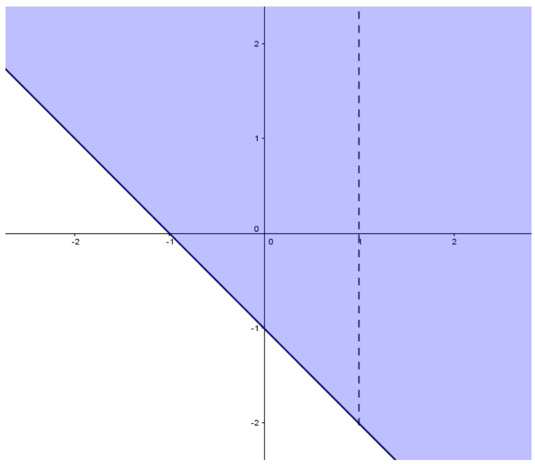
1. $\sqrt{x + y + 1}$

   We need to make sure $x + y + 1 > 0$. Rewrite it as $y > -1 - x$.

2. $x - 1$

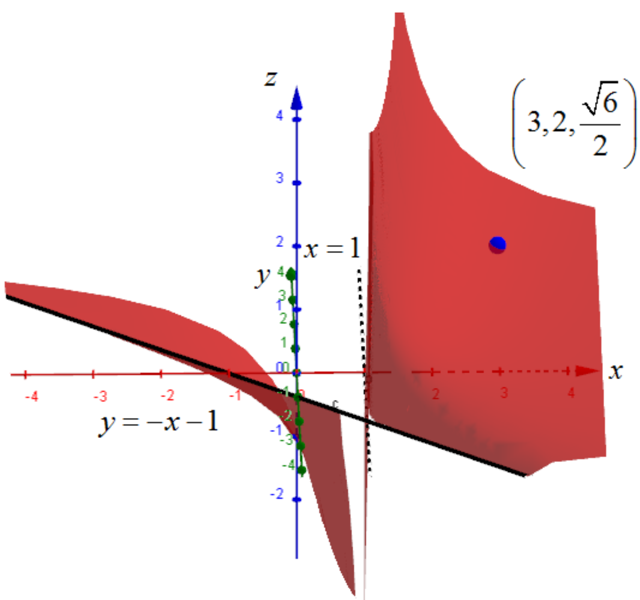
   We need to make sure the denominator $x - 1 \neq 0$, which happens at $x = 1$.

3. Putting this together we have $D = \{(x, y) \in \mathbb{R} | y \geq -x - 1, x \neq 1\}$. Here's a sketch of the domain on the xy-plane.



4. $f(3, 2) = \sqrt{3 + 2 + 1}3 - 1 = \sqrt{6}2$

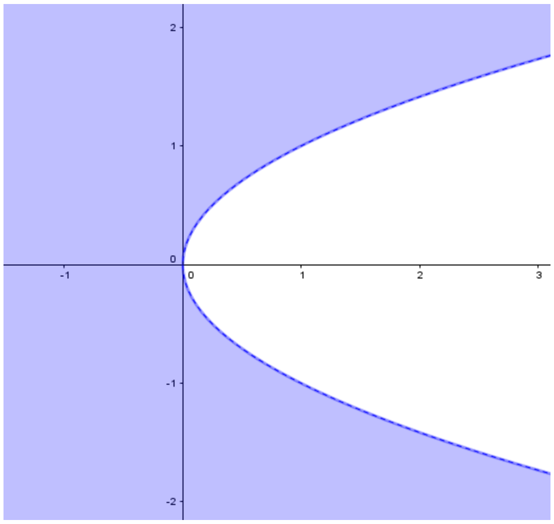
5. Sketching the original function with our domain:



Example 3

Find the domain of $f(x, y) = x \ln(y^2 - x)$ and $f(3, 2)$.

1. $\ln(y^2 - x) \Rightarrow y^2 - x > 0$

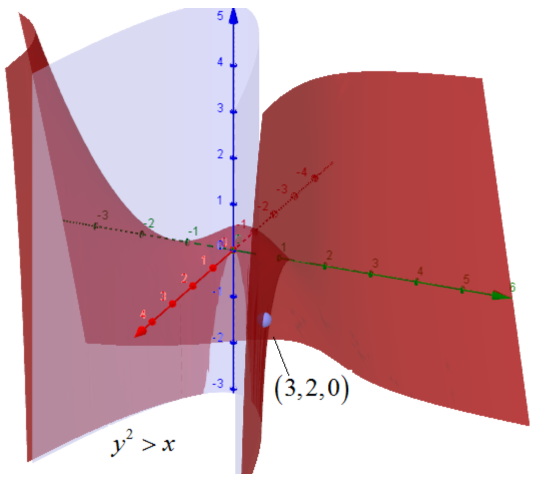


$$y^2 - x > 0$$

$$y^2 > x$$

$$D = \{(x, y) \in \mathbb{R}^2 | y^2 > x\}$$

2. Sketch



$$f(3, 2) = 3\ln(2^2 - 3) = 3\ln(1) = 0$$

Example 4

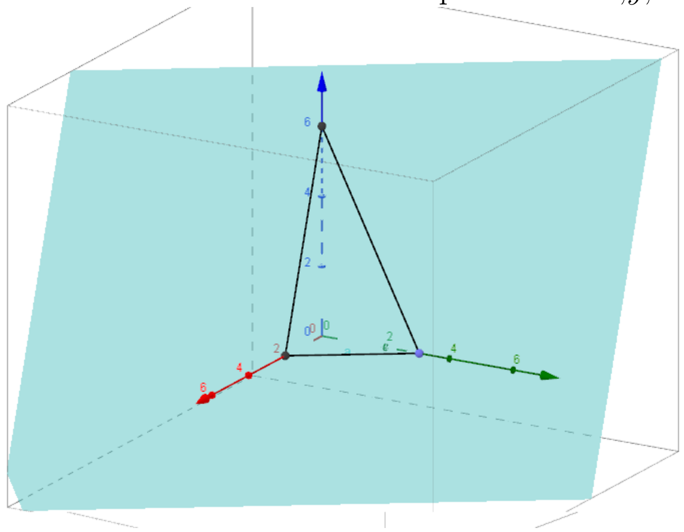
Sketch $f(x, y) = 6 - 3x - 2y$

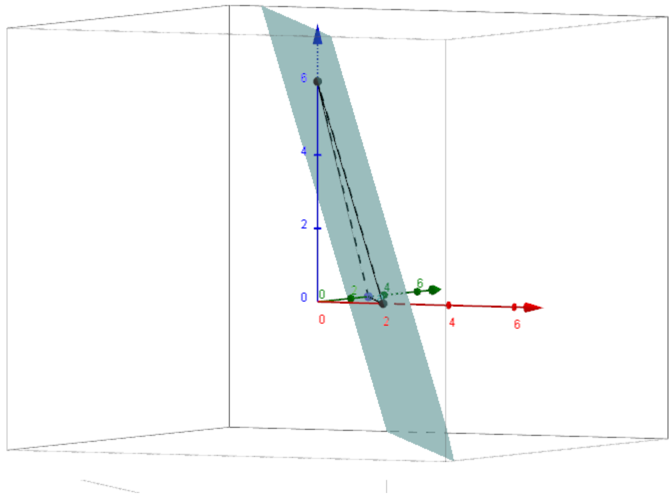
There are many different ways to try sketching a three dimensional surface. Most times a computer is used (like all my graphs). In this case the technique is to rewrite the function as

$$z = 6 - 3x - 2y$$

$$3x + 2y + z = 6$$

which is the equation of a plane with normal vector $< 3, 2, 1 >$. I sketched the plane from two different views. In both cases I plotted the x,y, and z intercepts.



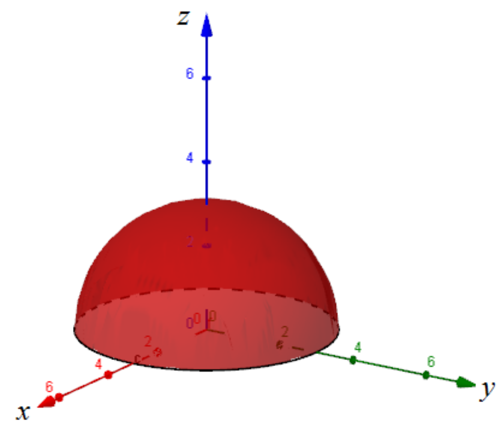


Example 5: Using Contour Plots (level curves)

In this example we will sketch $f(x, y) = \sqrt{9 - x^2 - y^2}$ by trying to

1. Rewrite it as a function we recognize.
2. Contour Plots (level curves).

1. Rewrite $z = \sqrt{9 - x^2 - y^2}$ as $x^2 + y^2 + z^2 = 9$. This is similar to a cirlce in 2D $x^2 + y^2 = 9$. In 3D it would be a sphere when you add z^2 with radius $r = 3$. Because the original function started with a $\sqrt{}$, we only take the positive z-axis.

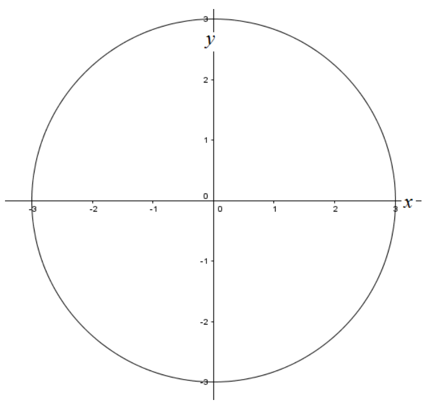


2. Next we'll do it with contour plots (level curves).

Definition 2: Level Curves

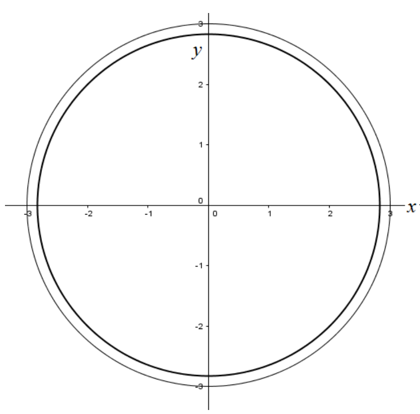
The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant. These curves are drawn in the 2D xy plane.

(a) Let $k = 0$:



$$0 = \sqrt{9 - x^2 - y^2} \Rightarrow x^2 + y^2 = 9$$

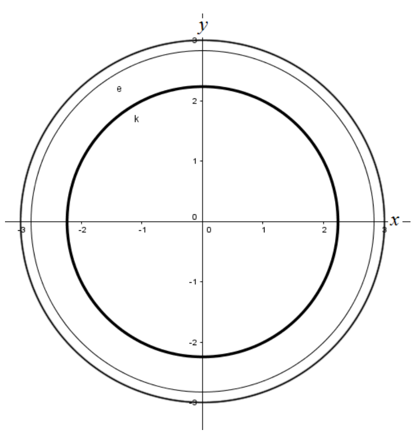
(b) Let $k = 1$



New level curve is bolded.

$$1 = \sqrt{9 - x^2 - y^2} \Rightarrow x^2 + y^2 = 8$$

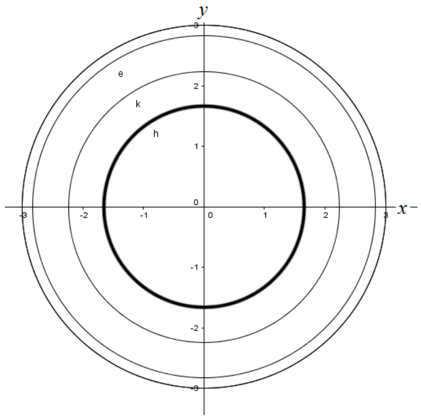
(c) Let $k = 2$



New level curve is bolded.

$$2 = \sqrt{9 - x^2 - y^2} \Rightarrow x^2 + y^2 = 5$$

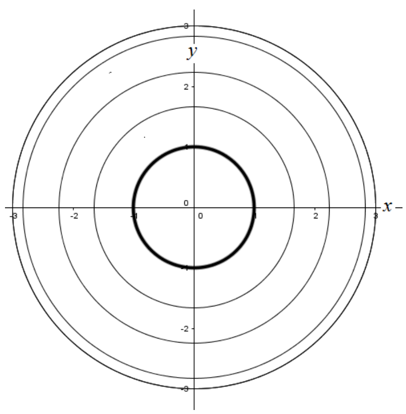
(d) Let $k = 2.5$



New level curve is bolded.

$$2.5 = \sqrt{9 - x^2 - y^2} \Rightarrow x^2 + y^2 = 2.75$$

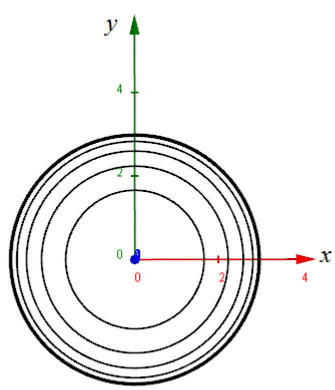
(e) Let $k = \sqrt{8} \approx 2.83$



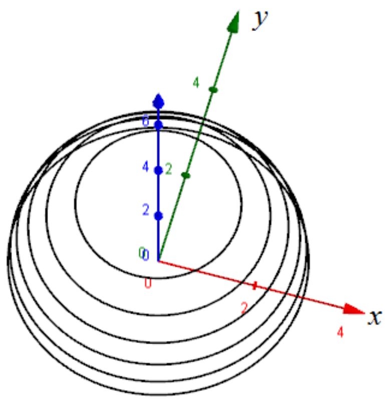
New level curve is bolded.

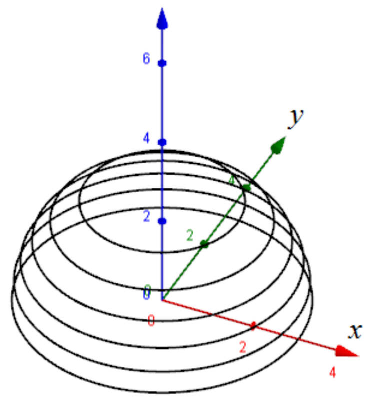
$$\sqrt{8} = \sqrt{9 - x^2 - y^2} \Rightarrow x^2 + y^2 = 1$$

Each line represents a height on the z-axis. If we look at a 3D graph but look straight down along the z-axis we get



Now let's start to rotate so we can see the z-axis.


y
x


y
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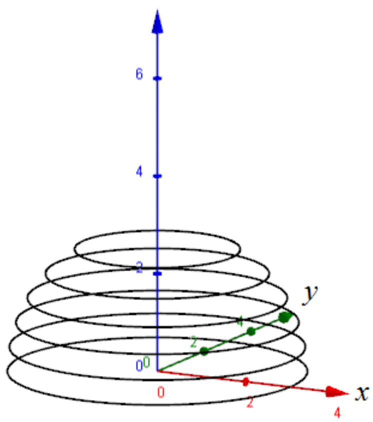
6
4
y
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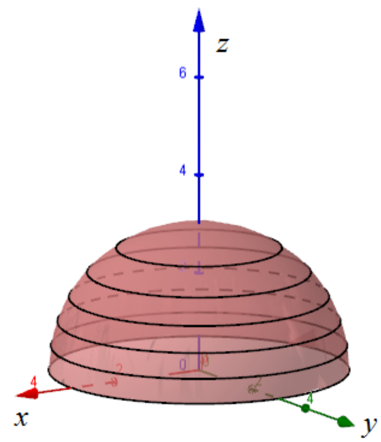
z
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