Math 232

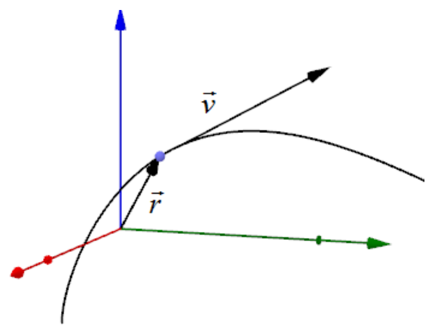
Calculus III

Brian Veitch • Fall 2015 • Northern Illinois University

13.4 Motion in Space

Definition 1

Let $\vec{r}(t)$ be the position vector.



$\vec{v}(t) = \vec{r'}(t)$ is the velocity vector and points in the diretion of the tangent vector.

The speed of the object at time t is the magnitude of $\vec{v}$.

$$\vec{s}(t) = |\vec{v}(t)|$$

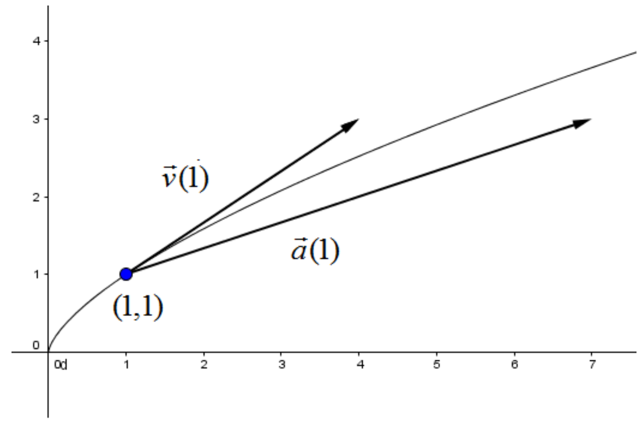
The acceleration of the object at time t is

$$\vec{a}(t) = \vec{v'}(t) = \vec{r''}(t)$$

Example 1

Position vector of an object moving is given by $\vec{r}(t) = t^3 i + t^2 j$. Find its velocity, speed, and acceleration at time $t = 1$.

1. $\vec{v}(t) = 3t^2 i + 2tj =< 3t^2, 2t >$
2. $\vec{s}(t) = |\vec{v}()| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$
3. $\vec{a}(t) = 6ti + 2j =< 6t, 2 >$.
4. At $t = 1$



$$\vec{v}(1) =< 3, 2 >= 3i + 2j$$

$$\vec{s}(1) = \sqrt{9(1)^4 + 4(1)^2} = \sqrt{13}$$

$$\vec{a}(1) =< 6, 2 >= 6i + 2j$$

Example 2

Find the velocity and position vectors of a particle with initial position $\vec{r}(0) = j - 4k$ and $\vec{v}(0) = -k$ where $\vec{a}(t) = \sin(t)i + 2\cos(t)j + 6tk$.

We are starting with acceleration and need to work our way up to position through integration.

1. Find $\vec{v}(t)$

$$\vec{v}(t) = \int \vec{a}(t)\ dt = \int \sin(t)i + 2\cos(t)j + 6tk\ dt$$
$$= -\cos(t)i + 2\sin(t)j + 3t^2k + C$$

2. We need to find the vector constant C using the initial velocity $\vec{v}(0) = -k$.

$$\vec{v}(0) = -k = -\cos(0)i + 2\sin(0)j + 3(0)^2k + C$$

$$-k = -i + C$$

$$C = i - k$$

It follows that

$$\vec{v}(t) = (-\cos(t) + 1)i + 2\sin(t)j + (3t^2 - 1)k$$

3. Let's repeat the whole process to find $\vec{r}(t)$

$$\vec{r}(t) = \int \vec{v}\, dt = \int (-\cos(t) + 1)i + 2\sin(t)j + (3t^2 - 1)k\, dt$$

$$= (-\sin(t) + t)i - 2\cos(t)j + (t^3 - t)k + C$$

4. We need to find the vector constant C by using the initial position $\vec{r}(0) = j - 4k$.

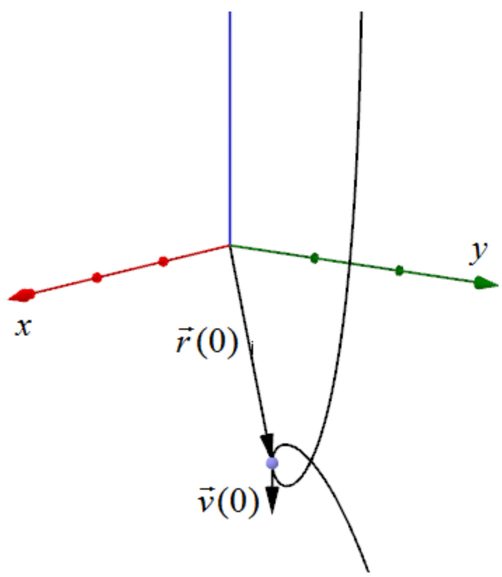
$$\vec{r}(0) = j - 4k = (-\sin(0) + 0)i - 2\cos(0)j + (0^3 - 0)k + C$$

$$j - 4k = -2j + C$$

$$C = 3j - 4k$$

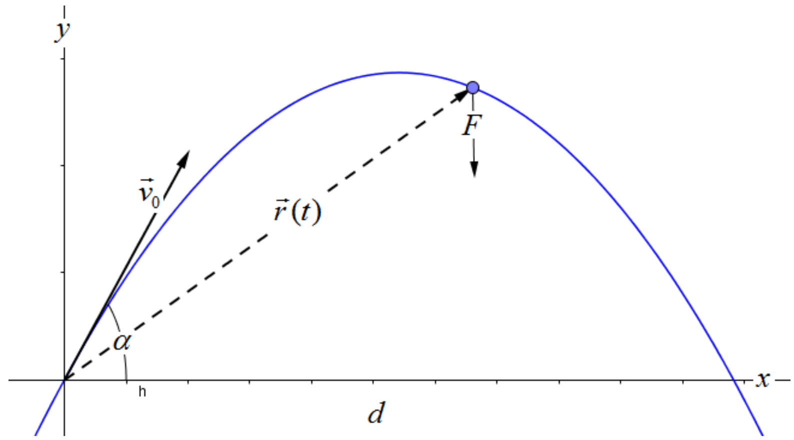
5. Final Answer:

$$\vec{r}(t) = (-\sin(t) + t)i + (-2\cos(t) + 3)j + (t^3 - t - 4)k$$



Projectile Motion

Suppose you have the following scenario:



$$\vec{r}_0 = \vec{r}(0) \text{ - initial position}$$

$$\vec{v}_0 = \vec{v}(0) \text{ - initial velocity}$$

Our goal is to come up with equations for the position function $\vec{r}(t)$. Let's start with the force F that acts on the object due to gravity.

$$F = ma = m(-gj) \text{ where } g \text{ is gravity}$$

meaning that $a(t) = -gj$. Now we start working our way back to $\vec{r}(t)$ through integration.

$$\vec{v}(t) = \int \vec{a}(t)\ dt = \int -gi\ dt$$
$$\vec{v}(t) = -gtj + C$$

To find C, plug in $t = 0$

$$\vec{v}(0) = \vec{v}_0 = -g(0)j + C$$

$$C = \vec{v}_0$$

This gives us $\vec{v}(t) = -gtj + \vec{v}_0$. Now to find $\vec{r}(t)$.

$$\vec{r}(t) = \int \vec{v}(t)\ dt = \int -gtj + \vec{v}_0\ dt$$

$$\vec{r}(t) = -\frac{1}{2}gt^2j + \vec{v}_0 t + D$$

To find D, plug in $t = 0$

$$\vec{r}(0) = \vec{r}_0 = D$$

So far $\vec{r}(t)$ is

$$\vec{r}(t) = -\frac{1}{2}gt^2 + \vec{v}_0 t + \vec{r}_0$$

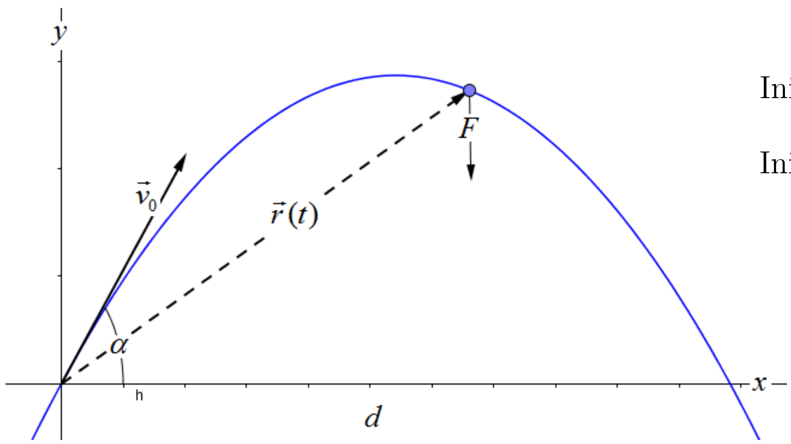
BUT $\vec{v}_0$ is made up of two components. You have the vertical velocity in terms of j and the horizontal velocity in terms of i.

$$\vec{v}_0 = (\vec{v}_0 \cos(\alpha))i + (\vec{v}_0 \sin(\alpha))j$$

Moving and combining like terms we finally have

$$\vec{r}(t) = (\vec{v}_0 \cos(\alpha))i + \left[\vec{r}_0 + \vec{v}_0 \sin(\alpha)t - \frac{1}{2}gt^2\right] j$$

Definition 2: Parametric Equations for Trajectory



Initial Position: $\vec{r}_0$

Initial Velocity: $\vec{v}_0$

$$\vec{r} = (\vec{v}_0 \cos(\alpha))i + \left[\vec{r}_0 + \vec{v}_0 \sin(\alpha)t - \frac{1}{2}gt^2\right] j$$

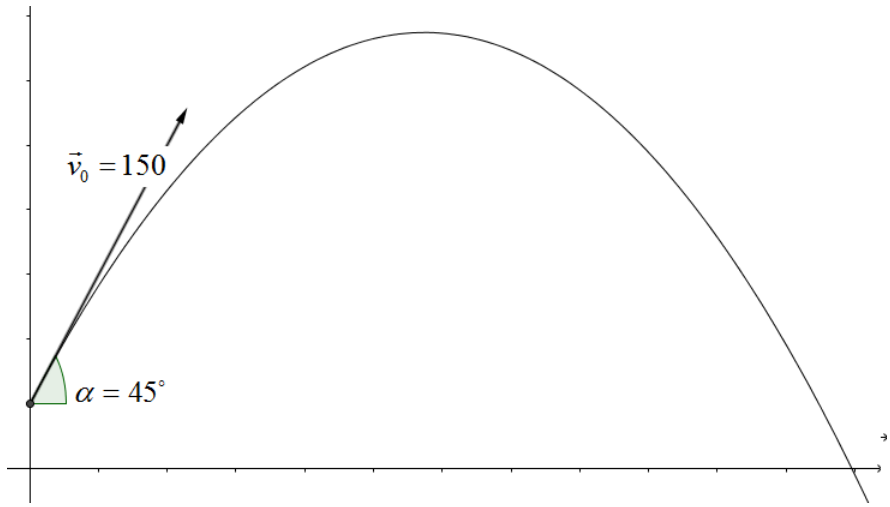
Horizontal Distance: $x(t) = (\vec{v}_0 \cos(\alpha))t$

Vertical Distance: $y(t) = \vec{r}_0 + (\vec{v}_0 \sin(\alpha))t - \frac{1}{2}gt^2$

Example 3

Projectile fired with initial velocity (muzzle speed) of 150 m/s with angle of elevation 45 degrees 10m above the ground. Where does it hit the ground? At what speed? What's the max height?

Let's do a quick sketch.



Let's start by setting up our trajectory equations. Use $g = 9.8$ for gravity.

$$\vec{r}(t) = 150\cos(\pi/4)ti + \left[10 + 150\sin(\pi/4)t - 4.9t^2\right] j$$

$$x(t) = 150\cos(\pi/4) = 75\sqrt{2}t$$

$$y(t) = 10 + 150\sin(\pi/4)t - 4.9t^2 = 10 + 75\sqrt{2} - 4.9t^2$$

To find out when the object hits the ground we first have to find for what value of t will the height equal 0.

1. Solve $y(t) = 0$

$$y(t) = 0$$
$$-4.9t^2 + 75\sqrt{2}t + 10$$
$$t = \frac{-75\sqrt{2} \pm \sqrt{(75\sqrt{2})^2 - 4(-4.9)(10)}}{2(-4.9)}$$
$$t = -0.09 \text{ or } 21.74$$

The object hits the ground at $t = 21.74$ seconds.

2. To find out where it hit the ground is the same thing as finding how far the object traveled horizontally.

$$x(21.74) = 75\sqrt{2}(21.74) = 2305.87 \text{ meters}$$

3. At speed does the object hit the ground? The speed at $t = 21.74$ seconds is $s(21.74) = |\vec{v}(21.74)|$. So let's find $\vec{v}(t)$

$$\vec{v}(t) = (75\sqrt{2})i + \left[75\sqrt{2} - 9.8t\right] j$$

$$\vec{v}(21.74) = 75\sqrt{2}i + \left[75\sqrt{2} - 9.8(21.74)\right] = 75\sqrt{2}i - 106.99j$$

$$s(21.74) = |75\sqrt{2}i - 106.99j| = \sqrt{(75\sqrt{2})^2 + (-106.99)^2} \approx 150.65 \text{ m/s}$$

4. Lastly, let's find the max height. The object reaches max height when $y'(t) = 0$.

$$y'(t) = 75\sqrt{2} - 9.8t = 0$$

$$t = \frac{75\sqrt{2}}{9.8} \approx 10.82 \text{ seconds}$$

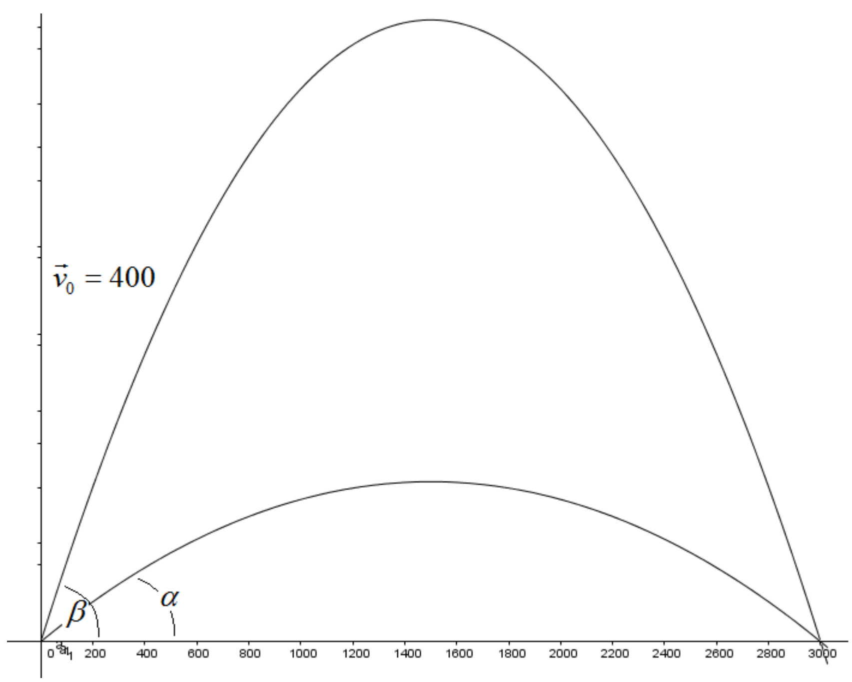
To get the max height plug $t = 10.82$ into $y(t)$

$$y(10.82) = 583.98 \text{ m}$$

Example 4

A projectile is fired with an initial speed of 400 m/s. At what two angles can be used to hit a target 3000m away?

Here's what we're dealing with:



I think this is a great opportunity solve a more general question. Given a distance d meters and an initial velocity $\vec{v}_0$, at what two angles can be used to hit the target? Solving the question with an arbitrary $\vec{v}_0$ means we can choose other velocities and get their corresponding angles.

1. Our goal is to get a relationship between α and $\vec{v}_0$ by removing the parameter t.

2. Let's write out our parametric equations for trajectory.

$$x = (\vec{v}_0)\cos(\alpha))t, \qquad y = -4.9t^2 + (\vec{v}_0 \sin(\alpha))t$$

3. Let T be time of impact.

$$d = (\vec{v}_0 \cos(\alpha))T$$

$$0 = -4.9t^2 + (\vec{v}_0 \sin(\alpha))t$$

4. Solve the first equation for T (it's easier).

$$T = \frac{d}{\vec{v}_0 \cos(\alpha)}$$

5. Plug $T = \dfrac{d}{\vec{v}_0 \cos(\alpha)}$ into $0 = -4.9t^2 + (\vec{v}_0 \sin(\alpha))t$

$$-4.9\left(\frac{d}{\vec{v}_0 \cos(\alpha)}\right)^2 + \vec{v}_0 \sin(\alpha)\left(\frac{d}{\vec{v}_0 \cos(\alpha)}\right) = 0$$

$$\frac{-4.9d^2}{\vec{v}_0^2 \cos^2(\alpha)} + d\tan(\alpha) = 0$$

$$\frac{-4.9d^2}{\vec{v}_0^2} + d\sin(\alpha)\cos(\alpha) = 0$$

$$\sin(\alpha)\cos(\alpha) = \frac{4.9d}{\vec{v}_0^2}$$

$$\frac{1}{2}\sin(2\alpha) = \frac{4.9d}{\vec{v}_0^2}$$

$$\sin(2\alpha) = \frac{9.8d}{\vec{v}_0^2}$$

At this point we know have a relationship between the angle, initial velocity, and distance. If you have two of these you can solve for the third. Since we're given distance and initial velocity we will keep solving for α.

$$2\alpha = \sin^{-1}\left(\frac{9.8d}{\vec{v}_0^2}\right)$$

Because of the way $\sin^{-1}(\theta)$ works, you need to actually solve two equations:

$$2\alpha = \sin^{-1}\left(\frac{9.8d}{\vec{v}_0^2}\right) \qquad\qquad 2\beta = \pi - \sin^{-1}\left(\frac{9.8d}{\vec{v}_0^2}\right)$$

Let $d = 3000$ and $\vec{v}_0 = 400$

$$\alpha = \frac{1}{2}\sin^{-1}\left(\frac{9.8 \cdot 3000}{400^2}\right)$$

$$\alpha = 0.0924 \text{ radians}$$

$$\alpha = 5.29 \text{ degrees}$$

$$2\beta = \pi - \sin^{-1}\left(\frac{9.8d}{\vec{v}_0^2}\right)$$

$$2\beta = \pi - 0.1848 = 2.9568$$

$$\beta = 0.1.4784 \text{ radians}$$

$$\beta = 84.706 \text{ degrees}$$