13.3 Arc Length

In section 10.2 with parametric equations \( x = f(t) \) and \( y = g(t) \), \( a \leq t \leq b \), the arc length was

\[
L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} \, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

The length of a space curve is defined in a similar way.

**Definition 1: Arc Length**

Let \( \vec{r}(t) = < f(t), g(t), h(t) > \) or \( x = f(t), y = g(t), z = h(t), a \leq t \leq b \), the length of the curve is

\[
L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt
\]

**Example 1**

Find the arc length of the circular helix with vector equations \( \vec{r}(t) = \cos(t)i + \sin(t)j + tk \) from \((1, 0, 0)\) to \((1, 0, 2\pi)\).

1. Since the bounds for \( t \) aren’t given we need to find them. What’s the value of \( t \) that gives us the points \((1, 0, 0)\) and \((1, 0, 2\pi)\).

\[
(1, 0, 0) \Rightarrow t = 0
\]

\[
(1, 0, 2\pi) \Rightarrow t = 2\pi
\]

2. Find \( \vec{r}' \)

\[
\vec{r}' = < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} >
\]
\[ \frac{dx}{dt} = -\sin(t) \]
\[ \frac{dy}{dt} = \cos(t) \]
\[ \frac{dz}{dt} = 1 \]

3. Use the Arc Length Formula

\[ L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \]
\[ = \int_{0}^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (1)^2} \, dt \]
\[ = \int_{0}^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1} \, dt \]
\[ = \int_{0}^{2\pi} \sqrt{2} \, dt \]
\[ = \sqrt{2t} \bigg|_{0}^{2\pi} \]
\[ = \sqrt{2}(2\pi) - \sqrt{2}(0) \]
\[ = 2\sqrt{2}\pi \]