Math 232

Calculus III

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13.3 Arc Length

In section 10.2 with parametric equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, the arc length was

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The length of a space curve is defined in a similar way.

Definition 1: Arc Length

Let $\vec{r}(t) =< f(t), g(t), h(t) >$ or $x = f(t)$, $y = g(t)$, $z = h(t)$, $a \leq t \leq b$, the length of the curve is

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example 1

Find the arc length of the circular helix with vector equations $\vec{r}(t) = \cos(t)i + \sin(t)j + tk$ from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

1. Since the bounds for t aren't given we need to find them. What's the value of t that gives us the points $(1, 0, 0)$ and $(1, 0, 2\pi)$.

$$(1, 0, 0) \Rightarrow \; t = 0$$

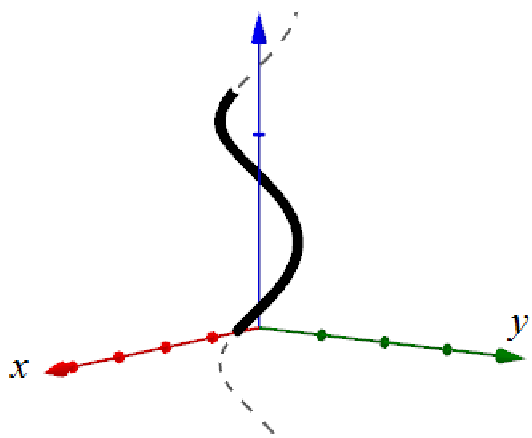
$$(1, 0, 2\pi) \Rightarrow \; t = 2\pi$$

2. Find $\vec{r'}$

$$\vec{r'} =< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} >$$

$$\frac{dx}{dt} = -\sin(t)$$
$$\frac{dy}{dt} = \cos(t)$$
$$\frac{dz}{dt} = 1$$

3. Use the Arc Length Formula



$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (1)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt \\ &= \int_0^{2\pi} \sqrt{2}\, dt \\ &= \sqrt{2}t\Big|_0^{2\pi} \\ &= \sqrt{2}(2\pi) - \sqrt{2}(0) \\ &= 2\sqrt{2}\pi \end{aligned}$$