Math 232

Calculus III

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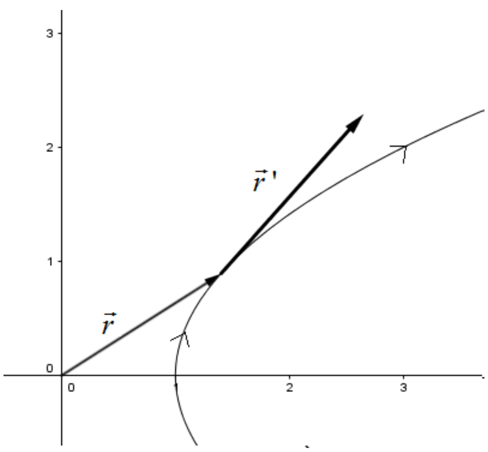
10.1 Derivative and Integrals of Vector Functions

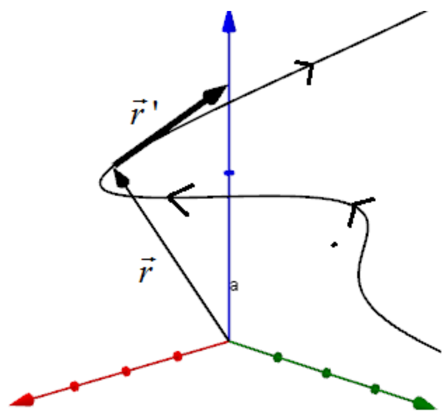
Definition 1: Derivative of a Vector Function

Given the vector function $\vec{r} =< f(t), g(t), h(t) >$, the derivative $\vec{r}'(t)$ is

$$\vec{r}'(t) =< f'(t), g'(t), h'(t) >$$

$$\vec{r}'(t) = f'(t)i + g'(t)j + h'(t)k$$





If $\vec{r'}$ is the tangent vector then $\dfrac{\vec{r'}}{|\vec{r'}|}$ is **Unit Tangent Vector**.

Example 1

Find $\vec{r}'(t)$, the unit tangent vector, and the tangent line at $t = 0$ when $\vec{r} = (1 + t^3)i + te^{-t}j + \sin(2t)k$.

1. $\vec{r}'(t) = 3t^2 i + (e^{-t} - te^{-t})j + 2\cos(2t)k$

2. Tangent Vector at $t = 0$ is

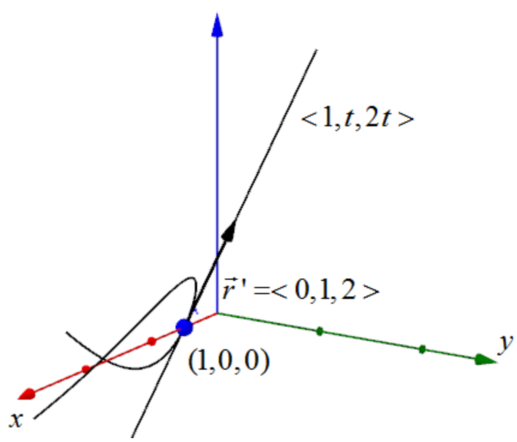
$$\vec{r'} = 0i + 1j + 2k =< 0, 1, 2 >$$

3. The Unit Tangent Vector at $t = 0$

$$\frac{\vec{r'}}{|\vec{r'}|} = \frac{<0,1,2>}{\sqrt{5}} =< 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} >$$

4. To find the tangent line we need a point and a direction. I'll use the point (1,0,0). I found this by evaluating $r(0)$. The direction vector is the tangent vector $< 0, 1, 2 >$.

$$L =< 1 + 0t, 0 + 1t, 0 + 2t >=< 1, t, 2t >$$



Example 2

For $\vec{r} = \sqrt{t}i + (2-t)j$, find $\vec{r'}$ and sketch the position vector $\vec{r}(1)$ and its tangent vector $\vec{r'}(1)$.

1. Find $\vec{r'}$

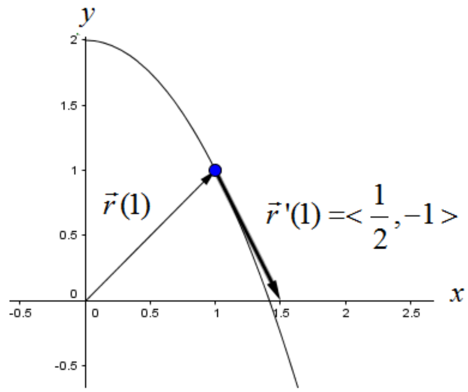
$$\vec{r'} = \frac{1}{2\sqrt{t}}i - 1j$$

2. The tangent vector is

$$\vec{r'}(1) = \frac{1}{2}i - j$$

3. The length is

$$\sqrt{(1/2)^2 + (-1)^2} = \frac{5}{2}$$



Example 3

Find the parametric equations for the tangent line of $x = 2\cos(t)$, $y = \sin(t)$, and $z = t$ at $(0, 1, \pi/2)$.

1. For what value of t gives us the point $(0, 1, \pi/2)$.

$$t = \pi/2$$

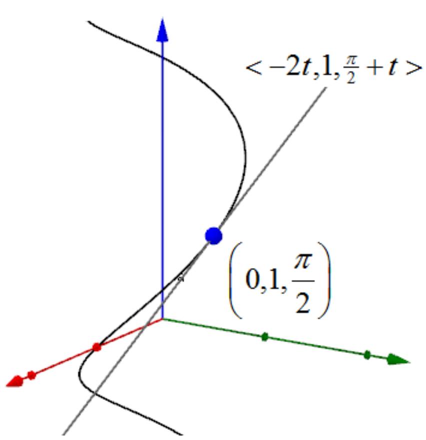
2. Find $\vec{r'}$

$$\vec{r'}(t) =< -2\sin(t), \cos(t), 1 >$$

3. Find the tangent vector

$$\vec{r'}(\pi/2) =< -2, 0, 1 >$$

4. The parametric equations for the tangent line are



$$x = 0 - 2t = -2t$$

$$y = 1 + 0t = 1$$

$$z = \pi/2 + 1t$$

$$< -2t, 1, \pi/2 + t >$$

Theorem 1: Differentation Rules for Vector Functions

Suppose $\vec{u}$ and $\vec{v}$ are differentiable vector functions, c is a scalar, and f is a real valued function.

1. $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \vec{u'}(t) \pm \vec{v'}(t)$
2. $\frac{d}{dt}\left[c\vec{u}(t)\right] = c\vec{u'}(t)$
3. $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u'}(t)$
4. $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u'}(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v'}(t)$
5. $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u'}(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v'}(t)$
6. $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\vec{u'}(t)$

Integrals

Definition 2: Integration of Vector Functions

Let $\vec{r}(t) = f(t)i + g(t)j + h(t)k =< f(t), g(t), h(t) >$. Then

$$\int_a^b \vec{r}(t)\; dt = \left[\int_a^b f(t)\; dt\right] i + \left[\int_a^b g(t)\; dt\right] j + \left[\int_a^b h(t)\; dt\right] k$$

Example 4

If $\vec{r}(t) = (2\cos(t))i + (\sin t)j + (2t)k$. Find

1. $\int \vec{r}(t)\; dt$
2. $\int_0^{\pi/2} \vec{r}(t)\; dt$

1. $\int \vec{r}(t)\ dt$

$$\begin{aligned}\int \vec{r}(t)\ dt &= \left[\int 2\cos(t)\ dt\right]\ i + \left[\int \sin(t)\ dt\right]\ j + \left[\int 2t\ dt\right]\ k \\ &= 2\sin(t)i - \cos(t)j + t^2k + C\end{aligned}$$

where C is called the vector constant. $C =< c1, c_2, c_3 >= c_1 i + c_2 j + c_3 k$.

$$(2\sin(t) + c_1)i + (-\cos(t) + c_2)j + (t^2 + c_3)k$$

2. $\int_0^{\pi/2} \vec{r}(t)\ dt$

$$\begin{aligned}\int_0^{\pi/2} \vec{r}(t)\ dt &= \left[\int_0^{\pi/2} 2\cos(t)\ dt\right]\ i + \left[\int_0^{\pi/2} t\sin(t)\ dt\right]\ j + \left[\int_0^{\pi/2} 2t\ dt\right]\ k \\ &= 2\sin(t)|_0^{\pi/2}\ i - \cos(t)|_0^{\pi/2}\ j + t^2|_0^{\pi/2}\ k + C \\ &= 2i + 1j + \frac{\pi^2}{4}k\end{aligned}$$

Example 5

Let $\vec{r'}(t) = ti + e^t j - te^{t^2}k$, where $\vec{r}(0) =< 1, 2, 1 >= i + 2j + k$.

$$\vec{r}(t) = \int ti + e^t j - te^{t^2}k\ dt$$

Let's take this one integral at a time.

1. $\left[\int t\ dt\right] i = \frac{1}{2}t^2 i$
2. $\left[\int e^t\ dt\right] j = e^t j$
3. $\left[\int -te^{t^2}\ dt\right] k$. This looks like a good candidate for u-substitution.
   (a) Let $u = t^2$
   (b) $du = 2t\ dt \Rightarrow \frac{1}{2}\ du = t\ dt$

(c) Substitute

$$\left[\int -\frac{1}{2}e^u \, du\right] k = -\frac{1}{2}e^u k = -\frac{1}{2}e^{t^2} k$$

So $\int \vec{r'} \, dt = \frac{1}{2}t^2 i + e^t j - \frac{1}{2}e^{t^2} k + C$

4. Now we use $\vec{r}(0) = i + 2j + k$ to find C.

$$\vec{r}(0) = i + 2j + k = \frac{1}{2}(0)^2 i + e^0 j - \frac{1}{2}e^0 k + C$$

$$i + 2j + k = j - \frac{1}{2}k + C$$

$$C = i - j + \frac{3}{2}k$$

Final Answer: $\vec{r}(t) = \left(\frac{1}{2}t^2 + 1\right) i + (e^t - 1)j + \left(-\frac{1}{2}e^{t^2} + \frac{3}{2}\right) k$