Math 232

Calculus III

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13.1 Vector Functions and Space Curves

Definition 1: Vector Functions

vector function has the form

$$\vec{r}(t) =< f(t), g(t), h(t) >= f(t)i + g(t)j + h(t)k$$

where $f(t)$, $g(t)$, and $h(t)$ are called the component functions of $\vec{r}$.

The domain of $\vec{r}(t)$ are all the values of t that work for $f(t)$, $g(t)$, and $h(t)$.

Example 1

Let $\vec{r}(t) =< t^3, \ln(3-t), \sqrt{t} >$. Find the components of $\vec{r}$ and its domain.

The components of $\vec{r}$ are

$$f(t) = t^3$$

$$g(t) = \ln(3-t)$$

$$h(t) = \sqrt{t}$$

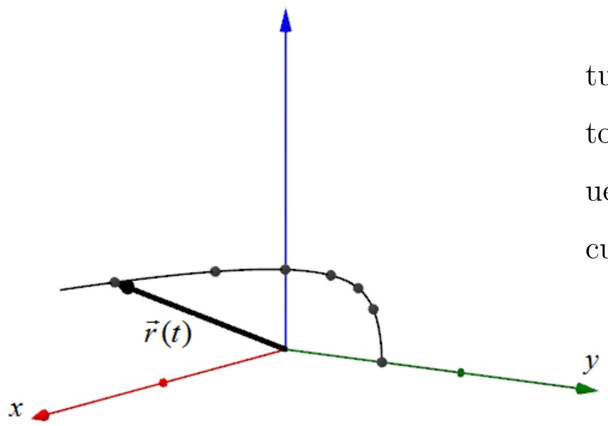
To find the domain let's find the domain of each component.

$$f(t) = t^3 \Rightarrow D = \{t | t \in \mathbb{R}\}$$

$$g(t) = \ln(3-t) \Rightarrow D = \{t | t < 3\} \text{ or } (-\infty, 3)$$

$$h(t) = \sqrt{t} \Rightarrow D = [0, \infty)$$

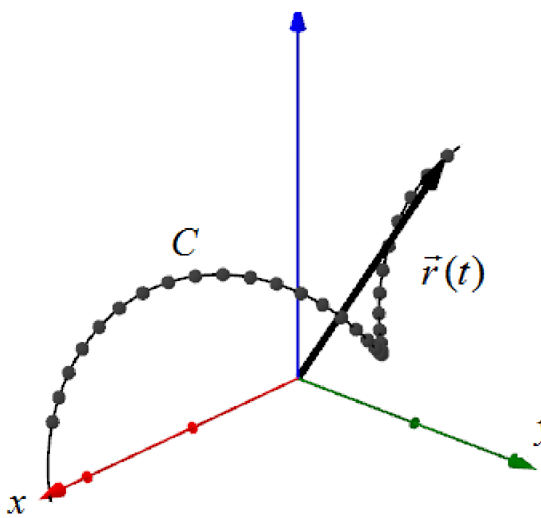
The values of t that work for all component functions is the domain $[0, 3)$.



Keep in mind that the vector function actually outputs vectors. However, if you were to trace the edge of the vector for all the values of t in the domain you would get a space curve.

Definition 2: Space Curves

Suppose f, g, and h are continuous functions on a domain D. Then the set of all points (x, y, z) in space where $x = f(t)$, $y = g(t)$, and $z = h(t)$ as t varies throughout D is called a Space Curve C.



C is traced out by the tip of the vectors from $\vec{r}(t)$.

Limits and Continuity

The limit of a vector $\vec{r}(t)$ is defined by taking the limit of its components.

$$\lim_{t\to a}\vec{r}(t) = <\lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)>$$

Example 2

Find the limit of $\vec{r}(t) =< \dfrac{1+t^2}{1-t^2}, \tan^{-1}(t), \dfrac{1-e^{-2t}}{t} >$ as $t \to \infty$.

1. $\lim\limits_{t\to\infty} \dfrac{1+t^2}{1-t^2} \overset{LH}{=} \lim\limits_{t\to\infty} \dfrac{2t}{-2t} = -1$
2. $\lim\limits_{t\to\infty} \tan^{-1}(t) = \pi/2$
3. $\lim\limits_{t\to\infty} \dfrac{1-e^{-2t}}{t} = \dfrac{1-0}{\infty} = 0$

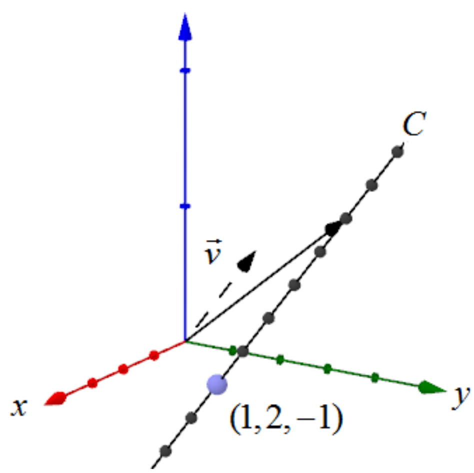
Final: $\lim\limits_{\to\infty} \vec{r}(t) =< -1, \pi/2, 0 >$

Example 3

Describe the curve defined by $\vec{r}(t) =< 1+t, 2+5t, -1+6t >$.

These are parametric equations for a line with direction vector $\vec{v} =< 1, 5, 6 >$ and initial point $(1, 2, -1)$.

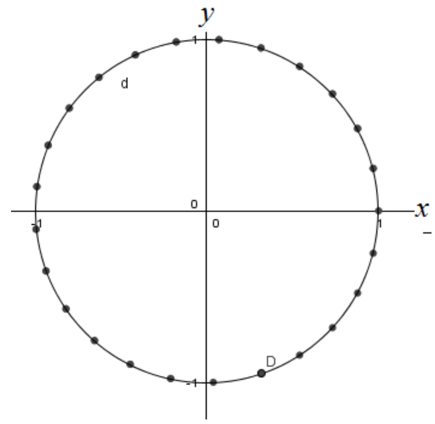
If you don't recognize it as a line then you can evaluate the vector function at various t values. I did this for a few values and got the following:



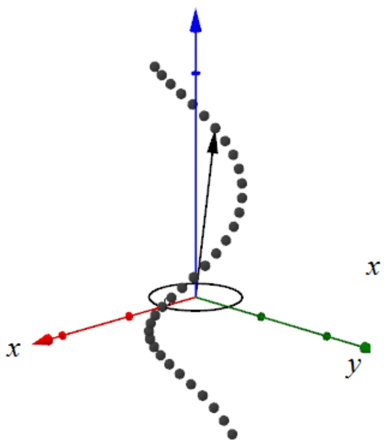
Example 4

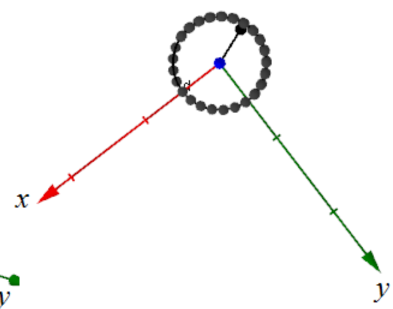
Sketch the curve defined by $\vec{r}(t) = \cos(t)i + \sin(t)j + tk$.

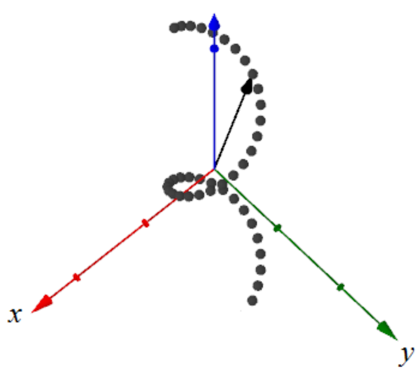
If you ignore the tk for a moment you get $x = \cos(t)$ and $y = \sin(t)$. In the xy plane you get a circle.



The $t\mathbf{k}$ part of the vector function just increases the value of z. I thought of it as extending a slinky. If you look straight down on it you only see a circle. But as you extend it you get the following:







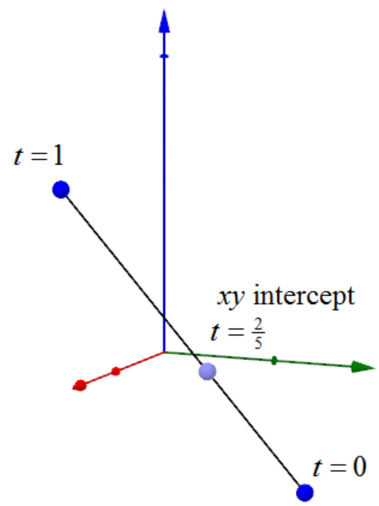
I rotated the image a few different ways so that you can get the full effect.

Example 5

Find vector and parametric equations for a line segment through points $P(1,3,-2)$ and $Q(2,-1,3)$.

This isn't any different than finding the equation of a line L. We need two things: a direction vector and a point.

1. $\vec{v} = \vec{PQ} =< 2-1, -1-3, 3+2 >=< 1, -4, 5 >$
2. Point: $P(1,3,-2)$.
3. Because this is a line segment we restrict the domain to $t \in [0,1]$.



Vector Equation:

$$\vec{r} =< 1+t, 3-4t, -2+5t >$$

Parametric Equations:

$$x = 1+t,\ y = 3-4t,\ z = -2+5t$$