Math 232

Calculus III

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12.3 The Dot Product

Definition 1: The Dot Product

Let $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$. Then the **Dot Product** is

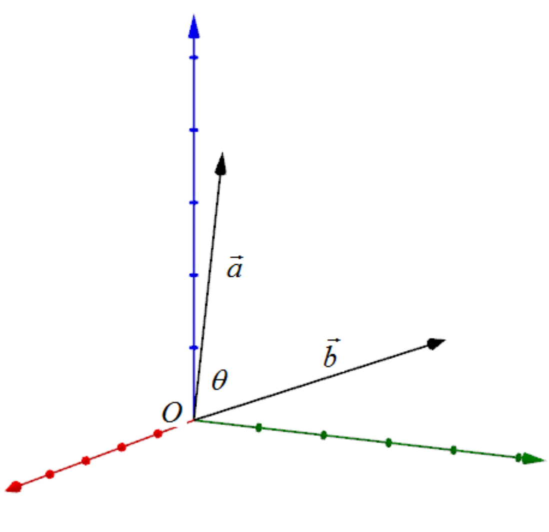
$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b + 3$$

Note: The dot product is a scalar (NOT ANOTHER VECTOR)

Properties of the Dot Product

1. $\vec{a} \cdot \vec{b} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{v}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{v}$
4. $0 \cdot \vec{a} = 0$ (the zero vector)
5. $(c\vec{a}) \cdot \vec{b} = c\left(\vec{a} \cdot \vec{b}\right)$

Theorem 1

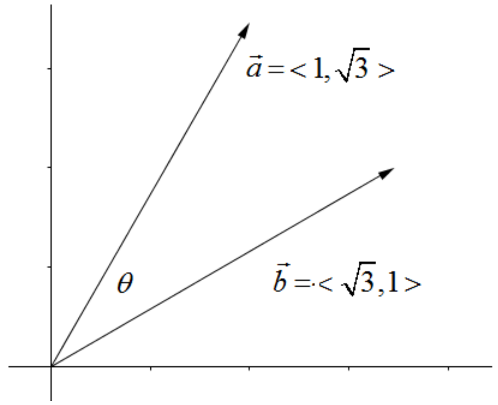


Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$. Then

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta) \text{ or } \cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Example 1

Find the angle between the vectors $\vec{a} =< 1, \sqrt{3} >$ and $\vec{b} =< \sqrt{3}, 1 >$.



$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

1. Find the dot product $\vec{a} \cdot \vec{b}$

$$\vec{a} \cdot \vec{b} =< 1, \sqrt{3} > \cdot < \sqrt{3}, 1 >= 1\sqrt{3} + \sqrt{3}1 = 2\sqrt{3}$$

2. Find $|\vec{a}|$ and $|\vec{b}|$

$$|\vec{a}| = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$|\vec{b}| = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$$

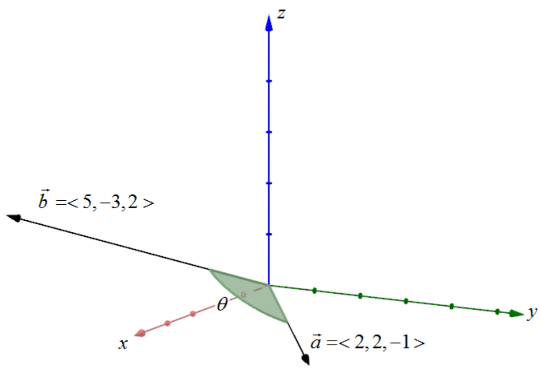
3. Use the formula $\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$$\cos(\theta) = \frac{2\sqrt{3}}{(2)(2)} = \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi/6 \ \ (30 \text{ degrees})$$

Example 2

Find the angle between the vectors $\vec{a} =< 2, 2, -1 >$ and $\vec{b} =< 5, -3, 2 >$.



$$\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{10-6-2}{\sqrt{938}} = \frac{2}{3\sqrt{38}}$$

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx 84 \text{ degrees}$$

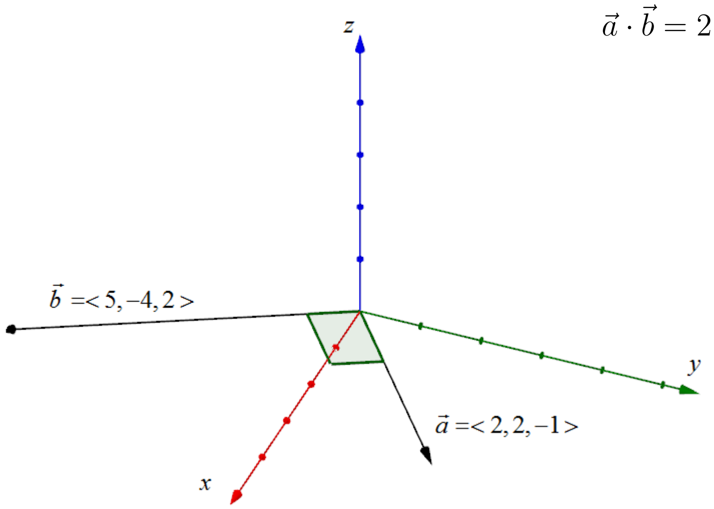
Definition 2: Orthogonal

Vectors $\vec{a}$ and $\vec{b}$ are Orthogonal or Perpendicular if $\vec{a}\cdot\vec{b} = 0$

If $\vec{a}\cdot\vec{b} > 0$, the angle is acute.
If $\vec{a}\cdot\vec{b} = 0$, the angle is right.
If $\vec{a}\cdot\vec{b} < 0$, the angle is obtuse.

Example 3

Let $\vec{a} =< 2, 2-1 >$ and $\vec{b} =< 5, -4, 2 >$. Show $\vec{a}$ and $\vec{b}$ are orthogonal.

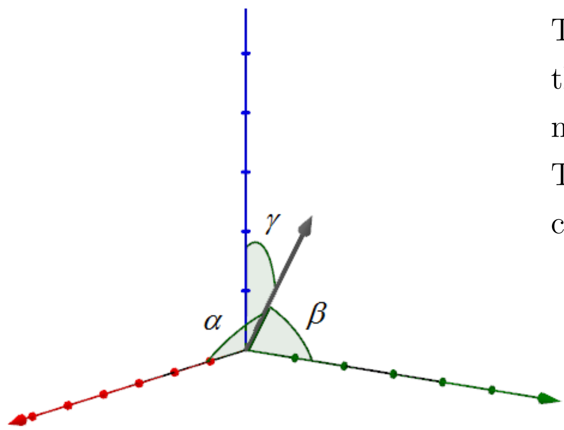


$\vec{a}\cdot\vec{b} = 2(5) + 2(-4) - 1(2) = 0$ orthogonal

Direction Cosines and Direction Angles

Definition 3: Direction Cosines and Direction Angles

Consider the graph below:



The **Direction Angle** of a vector $\vec{a}$ are the angles α, β, and γ on $[0, \pi]$ that $\vec{a}$ makes with the positive x, y, z axes.
The **Direction Cosines** are $\cos(\alpha)$, $\cos(\beta)$, and $\cos(\gamma)$.

$$\cos(\alpha) = \frac{a_1}{|\vec{a}|} \Rightarrow a_1 = \vec{a}\cos(\alpha)$$
$$\cos(\beta) = \frac{a_2}{|\vec{a}|} \Rightarrow a_2 = \vec{a}\cos(\beta)$$
$$\cos(\gamma) = \frac{a_3}{|\vec{a}|} \Rightarrow a_3 = \vec{a}\cos(\gamma)$$

$$\vec{a} =< a_1, a_2, a_3 >=< |\vec{a}|\cos(\alpha), |\vec{a}|cos(\beta), |\vec{a}|cos(\gamma) >$$

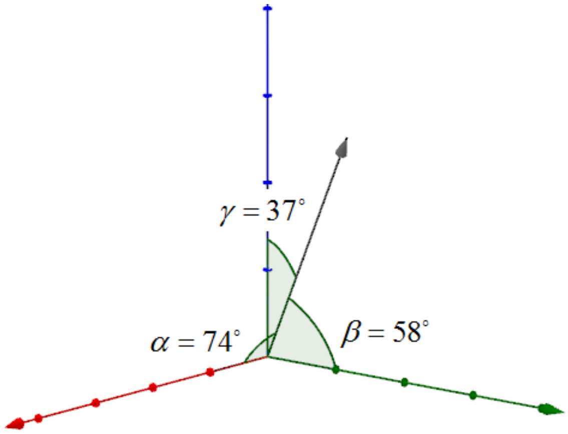
Also note that $\frac{\vec{a}}{|\vec{a}|} =< \cos(\alpha), \cos(\beta), \cos(\gamma) >$ is a unit vector in the direction of vector $\vec{a}$.

Example 4

Find the direction angles of $\vec{a} =< 1, 2, 3 >$

1. First thing to do is find $|\vec{a}|$
$$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$
2. To find the direction angles we need to solve for α, β, and γ using
$$\cos(\alpha) = \frac{1}{\sqrt{14}}, \quad \cos(\beta) = \frac{2}{\sqrt{14}}, \quad \cos(\gamma) = \frac{3}{\sqrt{14}}$$

3. Solving for the angles we get



$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 74 \text{ degrees}$$

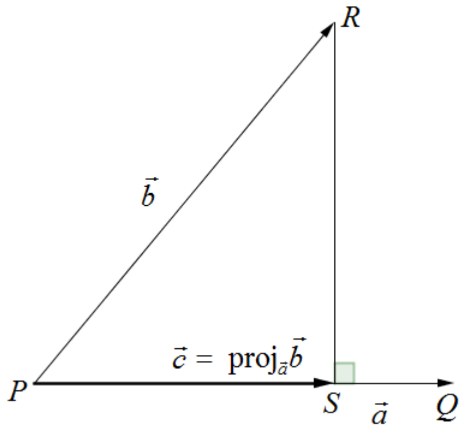
$$\beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58 \text{ degrees}$$

$$\gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37 \text{ degrees}$$

Projections

Definition 4: Vector Projection of $\vec{b}$ onto $\vec{a}$

It's much easier to visualize a vector projection in 2D than 3D. Let $\vec{a} = \vec{PQ}$, $\vec{b} = \vec{PR}$, and $\vec{c} = \vec{PS}$. Vector $\vec{c}$ is called the vector projection of $\vec{b}$ onto $\vec{a}$. Think of vector $\vec{c}$ as the shadow of $\vec{b}$ on $\vec{a}$ if you shined a light straight down over $\vec{b}$.



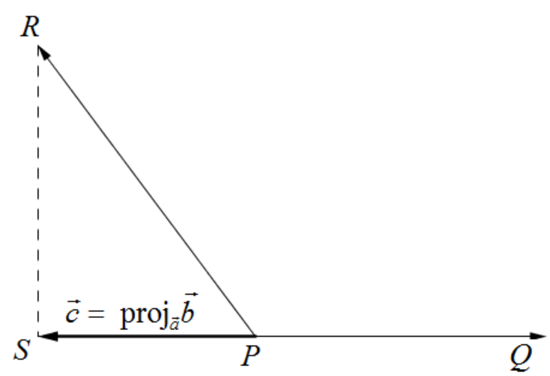
Vector Projection of $\vec{b}$ onto $\vec{a}$:

$$\vec{c} = \text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Scalar Projection of $\vec{b}$ onto $\vec{a}$

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

You can think of comp$_{\vec{a}}\vec{b}$ as the length of $\vec{c}$ with a $\pm$ to determine direction.
If the angle between vectors $\vec{a}$ and $\vec{b}$ is greater than 90 degrees, the picture would look like this:

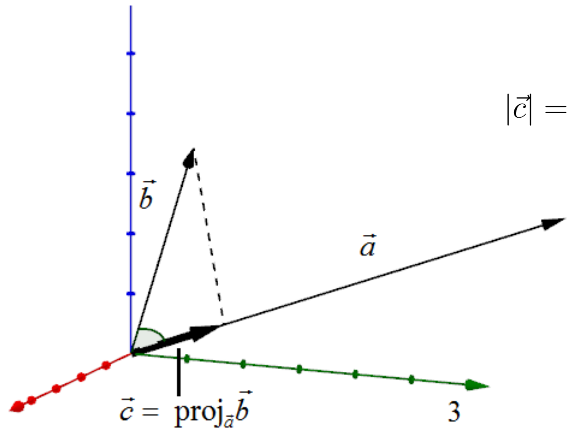


In the above graph comp$_{\vec{a}}\vec{b}$ is negative.

Example 5

Find the scalar and vector projection of $\vec{b} =< 1, 1, 2 >$ onto $\vec{a} =< -2, 3, 1 >$

Let me show you the vector $\vec{c}$ we are trying to find.



$$|\vec{c}| = \text{comp}_{\vec{a}}\vec{b} = \frac{< -2,3,1 > \cdot < 1,1,2 >}{\sqrt{(-2)^2 + (3)^2 + (1)^2}} = \frac{3}{\sqrt{14}}$$

The signed length of $\vec{c}$ is the scalar projection of $\vec{b}$ onto $\vec{a}$.

$$\vec{c} = \text{proj}_{\vec{a}}\vec{b} = \left(\frac{< -2,3,1 > \cdot < 1,1,2 >}{(\sqrt{14})^2}\right) < 1,1,2 >$$

$$= \frac{3}{14} < 1,1,2 >$$

$$=< -\frac{6}{14}, \frac{9}{14}, \frac{3}{14} >$$