12.1 Three Dimensional Coordinate Systems

**Definition 1: Point**

A point $P(a, b, c)$ is to be understand as

\[
\begin{align*}
    a &= x - \text{coordinate} \\
    b &= y - \text{coordinate} \\
    c &= z - \text{coordinate}
\end{align*}
\]

If $x = 0 \Rightarrow yz$ plane. If $y = 0 \Rightarrow xz$ plane. If $z = 0 \Rightarrow xy$ plane.
Definition 2: 3D Coordinate System

We say $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z)|x, y, z \in \mathbb{R}\}$ is the set all ordered pairs is known as the 3D coordinate system.

An equation involving $x, y$ in $\mathbb{R}^2$ is a curve.
An equation involving $x, y, z$ in $\mathbb{R}^3$ is a surface.

Example 1

Plot $A(-4, 0, -1), B(3, 1, -5),$ and $C(2, 4, 6)$

Example 2

Sketch $z = 4, x = 2,$ and $y = -3$ in $\mathbb{R}^3$

1. $z = 4$
2. $x = 2$

3. $y = -3$
Formula 1: Distance and Midpoint Between Points

Given two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ the distance between the two points is

Distance: \[ |PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \]

Midpoint: \[ \text{Mid}(PQ) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right) \]

Example 3

Find the midpoint and distance between the points $P(2, -1, 7)$ and $Q(1, -3, 5)$

\[ |PQ| = \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} = \sqrt{1 + 4 + 4} = \sqrt{5} \]

\[ \text{Mid}(PQ) = \left( \frac{1 + 2}{2}, \frac{-3 - 1}{2}, \frac{7 + 5}{2} \right) = \left( \frac{3}{2}, -2, 6 \right) \]
Formula 2
Suppose you have three points $P$, $Q$, and $R$. How do you know if the three points form a straight line?

**Straight Line:** To form a straight line you need

$$|PQ| + |QR| = |PR|$$

where $|PR|$ is the largest of the three lengths.

**Right Triangle:** To form a right triangle you need

$$|PQ|^2 + |QR|^2 = |PR|^2$$

where $|PR|$ is the the largest of the three lengths.

Example 4: Sketching Various Surfaces

1. $-x + y = 5, z = 0$
2. $-x + y = 5$
3. $x^2 + y^2 = 9, z = 2$
4. $x^2 + y^2 = 9$

1. $-x + y = 5, z = 0$

The $z = 0$ means the curve stays on the $xy$ plane. First start by finding the $x$ and $y$ intercepts. They are $(-5,0,0)$ and $(0,5,0)$. 
2. $-x + y = 5$. There is no restriction on $z$ meaning $z$ can be anything. This turns the line from the previous example into a plane.

3. $x^2 + y^2 = 9$, $z = 2$. In the $xy$ plane this is a circle with radius 3. Shift the circle up to $z = 2$. 
4. \( x^2 + 9^2 = 9 \). Since there is no restriction on \( z \) the circle can extend up and down along the \( z \) axis to form a cylinder.

**Definition 3: Equation of a Sphere**

\[ x^2 + y^2 + z^2 = r^2 \]

\[ (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2 \]
Example 5

Find the equation of a sphere that passes through the origin and has center $(1, 2, 3)$

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = r^2$$

Since the origin $(0, 0, 0)$ is on the surface we know the distance between $(0, 0, 0)$ and $(1, 2, 3)$ is $r$

$$r = \sqrt{(0 - 1)^2 + (0 - 2)^2 + (3 - 0)^2}$$

$$r = \sqrt{14}$$

Final answer is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 14$