Math 232

Calculus III

Brian Veitch • Fall 2015 • Northern Illinois University

12.1 Three Dimensional Coordinate Systems

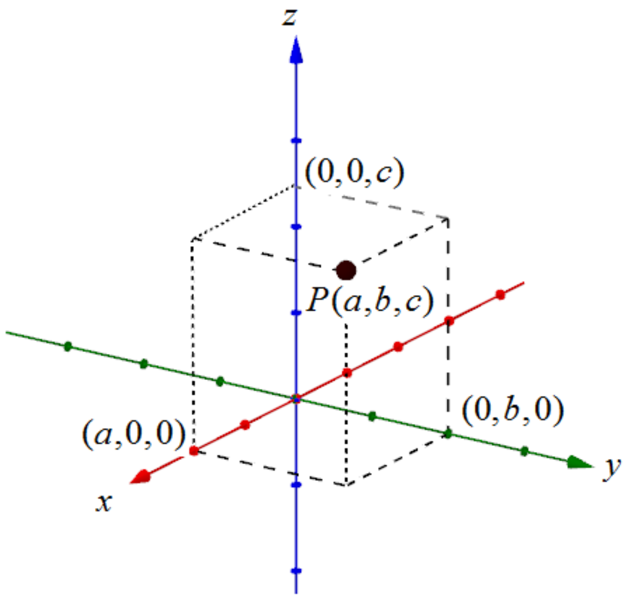
Definition 1: Point

A point $P(a,b,c)$ is to be understand as

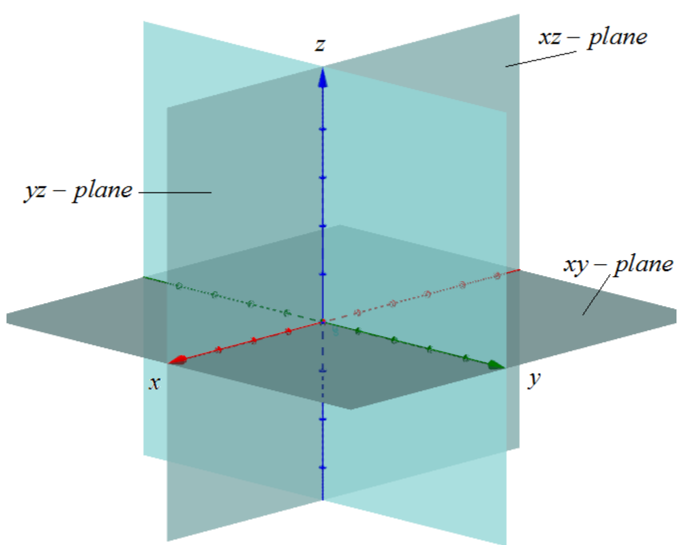
$$a = x - \text{ coordinate}$$

$$b = y - \text{ coordinate}$$

$$c = z - \text{ coordinate}$$



If $x = 0 \Rightarrow yz$ plane. If $y = 0 \Rightarrow xz$ plane. If $z = 0 \Rightarrow xy$ plane.



Definition 2: 3D Coordinate System

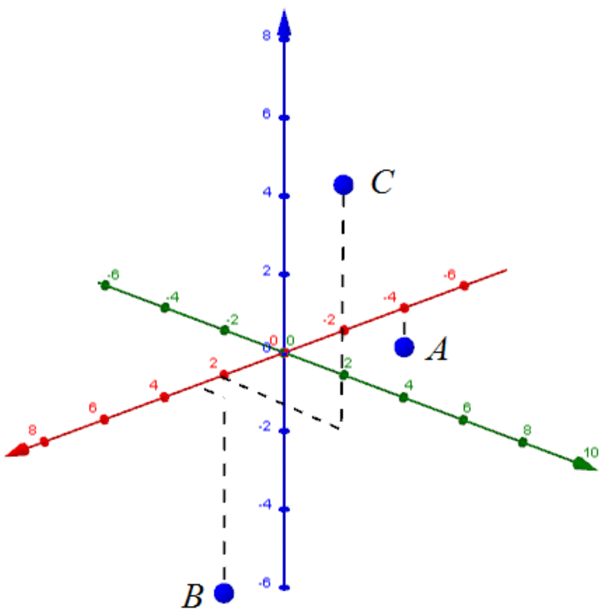
We say $\mathbb{R}^3 = \mathbb{R}\ x\ \mathbb{R}\ x\ \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set all ordered pairs is known as the 3D coordinate system.

An equation involving x, y in $\mathbb{R}^2$ is a **curve**.

An equation involving x, y, z in $\mathbb{R}^3$ is a **surface**.

Example 1

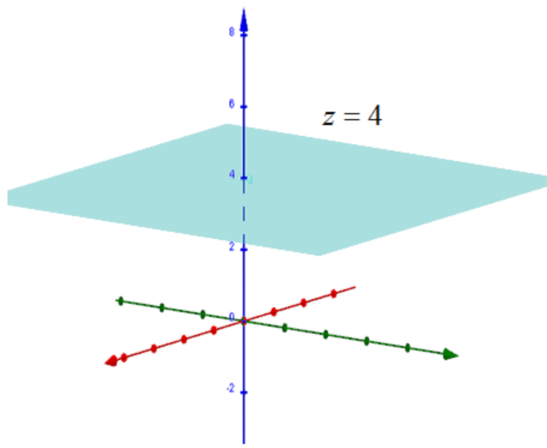
Plot $A(-4, 0, -1)$, $B(3, 1, -5)$, and $C(2, 4, 6)$



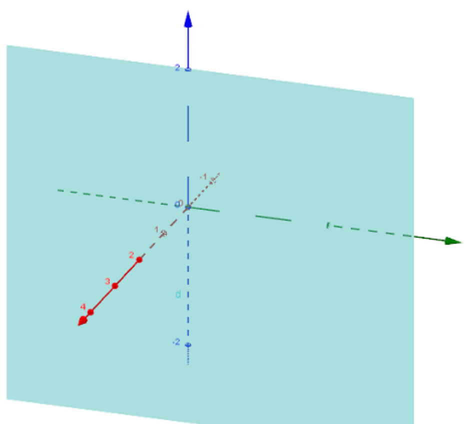
Example 2

Sketch $z = 4$, $x = 2$, and $y = -3$ in $\mathbb{R}^3$

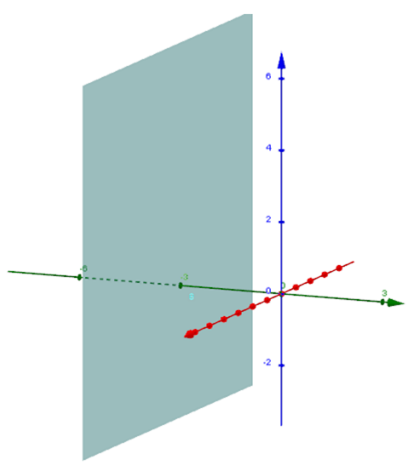
1. $z = 4$



2. $x = 2$



3. $y = -3$



Formula 1: Distance and Midpoint Between Points

Given two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ the distance between the two points is

$$\text{Distance: } |PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

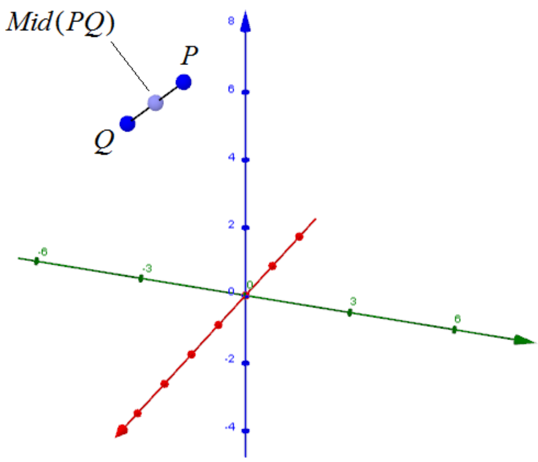
$$\text{Midpoint: } Mid(PQ) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example 3

Find the midpoint and distance between the points $P(2, -1, 7)$ and $Q(1, -3, 5)$

$$|PQ| = \sqrt{(1-2)^2 + (-3+1)^2 + (5-7)^2} = \sqrt{1+4+4} = \sqrt{5}$$

$$Mid(PQ) = \left(\frac{1+2}{2}, \frac{-3-1}{2}, \frac{7+5}{2}\right) = \left(\frac{3}{2}, -2, 6\right)$$



Formula 2

Suppose you have three points P, Q, and R. How do you know if the three points form a straight line?

Straight Line: To form a straight line you need

$$|PQ| + |QR| = |PR|$$

where $|PR|$ is the largest of the three lenghts.

Right Triangle: To form a right triangle you need

$$|PQ|^2 + |QR|^2 = |PR|^2$$

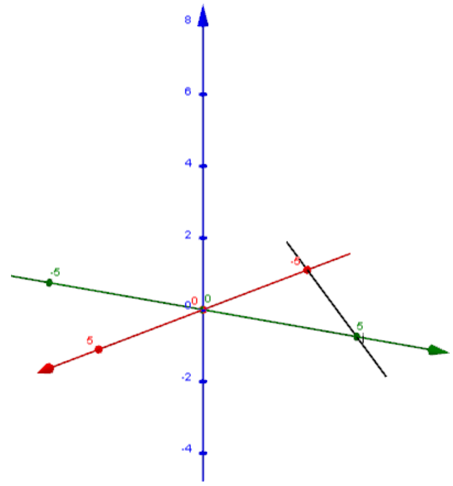
where $|PR|$ is the the largest of the three lengths.

Example 4: Sketching Various Surfaces

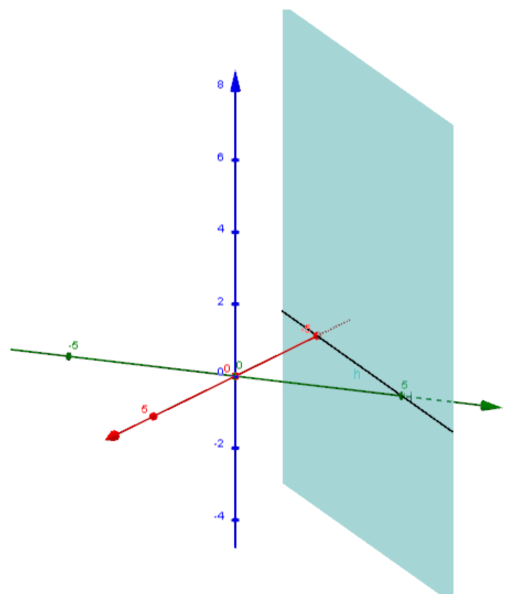
1. $-x + y = 5, z = 0$
2. $-x + y = 5$
3. $x^2 + y^2 = 9, z = 2$
4. $x^2 + y^2 = 9$

1. $-x + y = 5,\ z = 0$

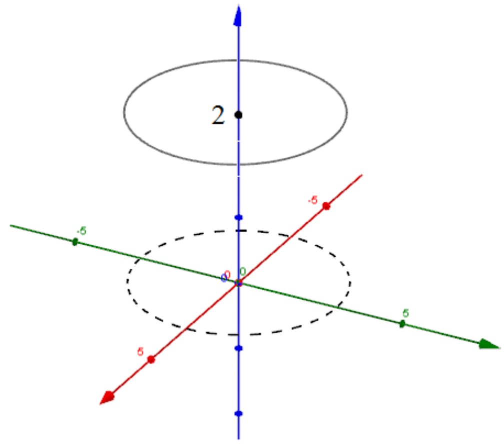
   The $z = 0$ means the curve stays on the xy plane. First start by finding the x and y intercepts. They are $(-5, 0, 0)$ and $(0, 5, 0)$.



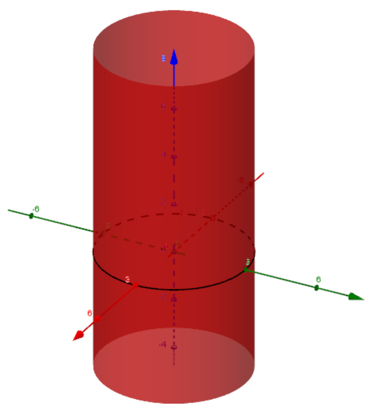
2. $-x + y = 5$. There is no restriction on z meaning z can be anything. This turns the line from the previous example into a plane.



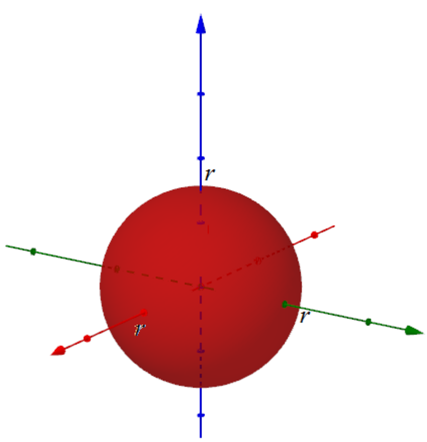
3. $x^2 + y^2 = 9$, $z = 2$. In the xy plane this is a circle with radius 3. Shift the circle up to $z = 2$.



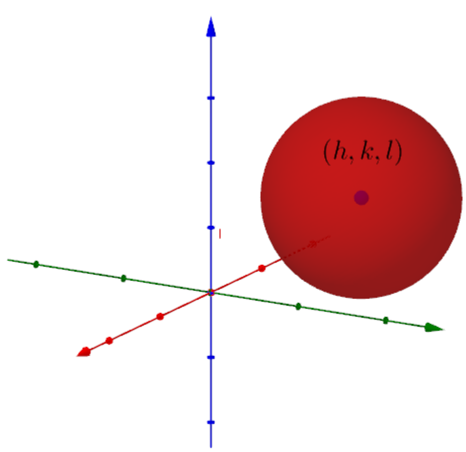
4. $x^2 + 9^2 = 9$. Since there is no restriction on z the circle can extend up and down along the z axis to form a cylinder.



Definition 3: Equation of a Sphere



$$x^2 + y^2 + z^2 = r^2$$



$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Example 5

Find the equation of a sphere that passes through the origin and has center $(1, 2, 3)$

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = r^2$$

Since the origina $(0, 0, 0)$ is on the surface we know the distance between (0,0,0) and (1,2,3) is r

$$r = \sqrt{(0-1)^2 + (0-2)^2 + (3-0)^2}$$

$$r = \sqrt{14}$$

Final answer is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$