Math 232

Calculus III

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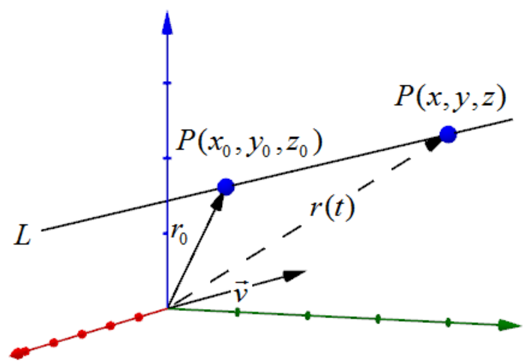
12.5 Equations of Lines and Planes

Definition 1: Vector Equation of a Line L



Let L be a line in three-dimensional space. $P(x, y, z)$ is an artibrary point on L. $P(x_0, y_0, z_0)$ is a specific point on L. r_0 is the vector that connects to $P(x_0, y_0, z_0)$ $r(t)$ is the vector that connects to a point on L. And $\vec{v}$ is the position vector that is parallel to L.

The vector equation for a line is three dimensions space is

$$\vec{r}(t) = \vec{v}t + \vec{r_0}$$

$$\vec{r}(t) =< a, b, c > t+ < x_0, y_0, z_0 >$$

where $\vec{v} =< a, b, c >$ and t is a parameter.

Definition 2: Parametric Equations of a Line L

Parametric equations for a line through point (x_0, y_0, z_0) and parallel to the direction vector $\vec{v} =< a, b, c >$ are

$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

Example 1

Find a vector equation and parametric equations for a line passing through the point (5,1,3) and is parallel to $i + 4j - 2k$. Find two additional points on this line.

We need two things: an initial point (x_0, y_0, z_0) and a direction vector parallel to the line L.

1. Our initial point is $(5, 1, 3)$ with initial vector $r_0 =< 5, 1, 3 >$
2. The direction vector is $\vec{v} = i + 4j - 2k$ or $< 1, 4, -2 >$

The vector equation is

$$\vec{r}(t) = \vec{v}t + r_0$$

$$\vec{r}(t) =< 1, 4, -2 > t+ < 5, 1, 3 >$$

$$\vec{r}(t) =< t + 5, 4t + 1, -2t + 3 >$$

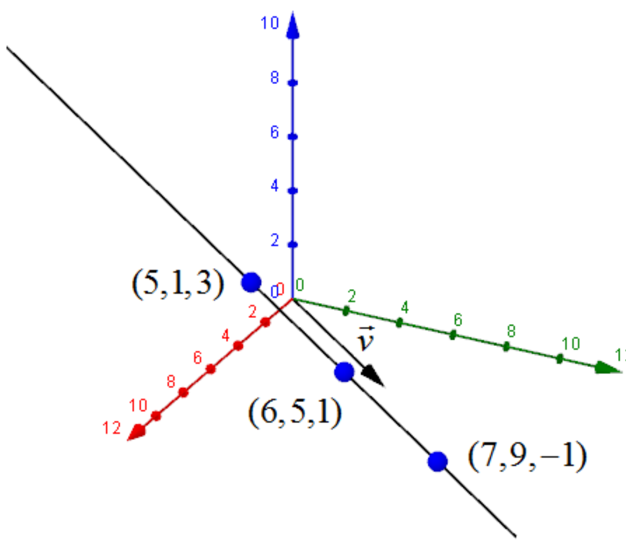
The parametric equations are

$$x = t + 5 \qquad y = 4t + 1 \qquad z = -2t + 3$$

To find two points on the line L choose two values of t different than $t = 0$.

$$t = 1 : \qquad P(6, 5, 1)$$

$$t = 2 : \qquad P(7, 9, -1)$$



It is possible to give an equation for the line L without using the parameter t. Consider the parametric equations for L

$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

Solve each one for t

$$t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

Since they all equal t, they must all equal each other.

Definition 3: Symmetric Equations of Line L

Let (x_0, y_0, z_0) be a point on L with direction vector $\vec{v} =< a, b, c >$. Then the symmetric equations for L are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 2

Find the parametric and symmetry equations for the line passing through $P(2, 4, -3)$ and $Q(3, -1, 1)$.

To find the equation of a line we need two things: an initial point, and a direction vector parallel to our line. We weren't given a direction vector; however we were given two points. Since the points are on the same line, the vector connecting the two points will be parallel to our line.

1. The direction vector $\vec{v}$ can be found by finding $\vec{PQ}$

$$\vec{PQ} =< 3 - 2, -1 - 4, 1 + 3 >=< 1, -5, 4 >$$

2. We can choose either point as our initial point. I'll choose $P(2, 4, -3)$.

   The vector equation is

$$\vec{r} =< 1, -5, 4 > t+ < 2, 4, -3 >=< t + 2, -5t + 4, 4t - 3 >$$

   The parametric equations are

$$x = t + 2 \qquad y = -5t + 4 \qquad z = 4t - 3$$

   The symmetric equation

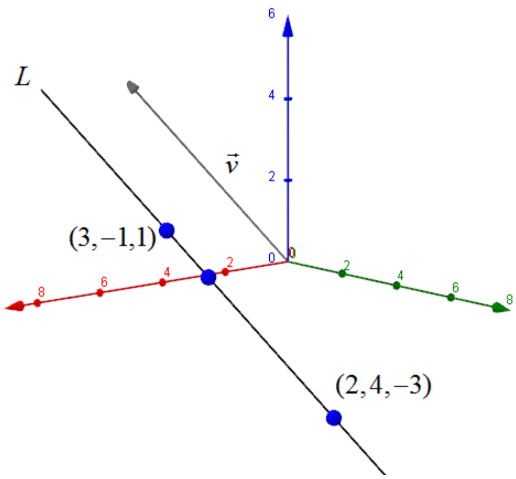
$$\frac{x - 1}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}$$

3. At what point does the line intersect the xy-plane?

Any point on the xy plane has a z-value of $z = 0$. Plug $z = 0$ into the symmetric equations to get two equations

$$\frac{x-1}{1} = \frac{(0)+3}{4} = \frac{3}{4}$$
$$\frac{y-4}{-5} = \frac{(0)+3}{4} = \frac{3}{4}$$

Solving the two equations for x and y respectively, we get $x = \frac{11}{4}$ and $y = \frac{1}{4}$.



L crosses the xy plane at the point $\left(\frac{11}{4}, \frac{1}{4}, 0\right)$.

Summary 1: How to Find the Direction Vector $\vec{v}$

1. You can find the direction vector $\vec{v} = < a, b, c >$ of any of the three equations for a line (vector, parametric, symemtric).

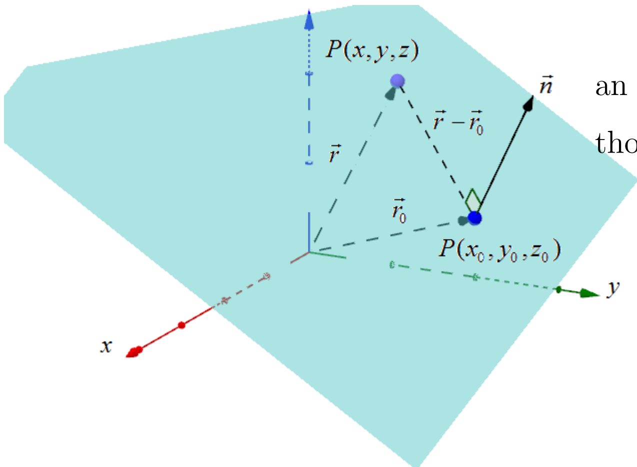
2. Parallel to another line L_2? Use the direction vector on L_2.

3. Given two points P and Q? Then your direction vector $\vec{v} = \vec{PQ}$

4. Remember that two lines are parallel if their direction vectors $\vec{v_1}$ and $\vec{v_2}$ are proportional.
$$\vec{v_1} = \lambda \vec{v_2}$$

5. Perpendicular to another line L_2? If L_2 has direction vector $\vec{v_2}$ then your direction vector $\vec{v_1}$ must satisfy
$$\vec{v_1} \cdot \vec{v_2} = 0$$

Planes



To create a plane you need two things: an initial point (x_0, y_0, z_0) and a vector $\vec{n}$ orthongonal to the plane.

From the graph above there is one vector shown that is on the plane. That vector is $\vec{r} - \vec{r_0}$. Since $\vec{n}$ is orthogonal to the plane it must be perpendicular to $\vec{r} - \vec{r_0}$. It follows that

Vector Equation of a Plane: $(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$

Let $\vec{n} =< a, b, c >$ be the normal (orthogonal) vector to the plane, $r =< x, y, z >$ be any vector going to the plane, and $r_0 =< x_0, y_0, z_0 >$ be the vector going to the initial point.

$$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$$

$$< x - x_0, y - y_0, z - z_0 > \cdot < a, b, c >= 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Definition 4: Planes

Let $P(x_0, y_0, z_0)$ be a point on a plane with normal vector $\vec{n} =< a, b, c >$. **Vector Equation of the Plane**

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

$$\vec{n}\cdot < x, y, z >= \vec{n}\cdot < x_0, y_0, z_0 >$$

Scalar Equation of the Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Example 3

Find the equation of the plane through (2,4,-1) with normal vector $\vec{n} =< 2, 3, 4 >$. Then sketch the plane.

We need two things for the equation of a plane: a point, and a normal vector. We have both already!

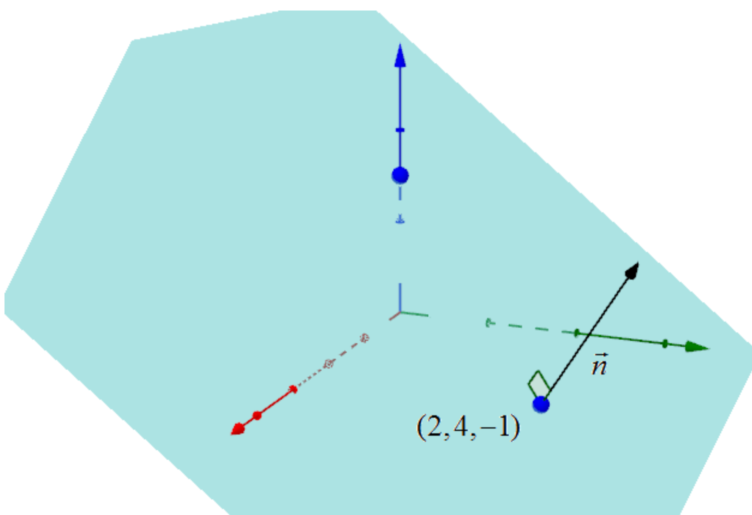
1. The point is (2,4,-1)
2. The normal vector $\vec{n} =< 2, 3, 4 >$
3. Using the scalar equation of a plane we get

$$2(x-2)+3(y-4)+4(z+1)=0$$

$$2x-4+3y-12+4z+4=0$$

$$2x+3y+4z=12$$

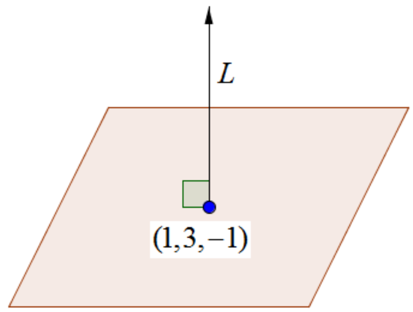
4. Sketch



Example 4

Find the equation of a plane through the point (1,3,-1) and perpedicular to the line $x = 2 - t$, $y = t + 1$, $z = 3 - 2t$.

I drew a quick sketch of the information I was given.



If the plane is perpendicular to a line L, then the direction vector of that line is also perpendicular to the plane. Let's find the direction vector.

1. To find the direction vector of the line $x = 2 - t$, $y = t + 1$, and $z = 3 - 2t$, we can use the general form of the parametric equations to extract $\vec{v} =< a, b, c >$.

   Recall that the parametric equations of a line are

   $$x = at + x_0, \qquad y = bt + y_0, \qquad z = ct + z_0$$

   Matching up our parametric equations with the general ones it follows that the direction vector is

   $$\vec{v} =< -1, 1, -2 >$$

2. Since we determined already that the direction vector of the line L is perpendicular to the plane we can use it as the normal vector $\vec{n}$.

   $$\vec{n} =< -1, 1, 2 >$$

3. Using the scalar equation of a plane we get

   $$-1(x - 1) + 1(y - 3) - 2(z + 1) = 0$$

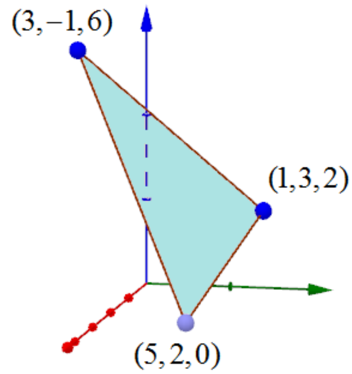
   $$-x + 1 + y - 3 - 2z - 2 = 0$$

   $$-x + y - 2z = 4$$

Example 5

Find an equation of the plane that passes through $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

A quick sketch of the graph we have the following:



To find the equation of the plane we need two things: a point, and the normal vector.

1. We choose any of the three points given. I'll use $P(1, 3, 2)$.

2. Finding the normal vector is a bit trickier. I mentioned this a few sections back but you can connect any two points on a plane to get a vector. If we take the cross product of those two vectors we will get a new vector perpendicular to not only the two original vectors but also the plane.

$$\vec{PQ} =< 3-1, -1-3, 6-2 >=< 2, -4, 4 >$$
$$\vec{PR} =< 5-1, 2-3, 0-2 >=< 4, -1, -2 >$$

$$\begin{aligned}\vec{n} = \vec{PQ} \times \vec{PR} &= \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} \\ &= \begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix} i - \begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix} j + \begin{vmatrix} 2 & -4 \\ 4 & -1 \end{vmatrix} k \\ &= 12i + 20j + 14k \end{aligned}$$

Therefore our normal vector is $\vec{n} =< 12, 20, 14 >$.

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

$$12x - 12 + 20y - 60 + 14z - 28 = 0$$

$$12x + 20y + 14z = 100$$

$$6x + 10y + 7z = 50$$

Example 6

Find a piont at which the line $x = 2 + 3t$, $y = -4t$, and $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

We can plug in $x = 2 + 3t$, $y = -4t$, and $z = 5 + t$ into $4x + 5y - 2z = 18$ and solve for t. Once we know t, we can find the specific point.

$$4(2 + 3t) + 5(-4t) - 2(5 + t) = 18$$

$$8 + 12t - 20t - 10 - 2t = 18$$

$$-10t - 2 = 18$$

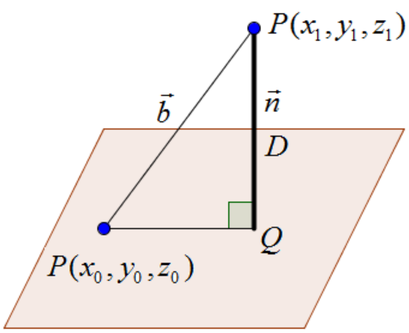
$$t = -2$$

Plugging $t = 2$ into the parametric equations for the line and we get

$$x = -4, \ y = 8, \ z = 3 \ \ (-4, 8, 3)$$

Distances

Suppose you want to find the closest distance from a point in three dimensional space $P(x, y, z)$ to a plane. We are looking for the distance D in the graph below.



The closest point will lie on a vector perpendicular to the plane. We can call this $\vec{n}$ since it's perpendicular to the plane. Let $\vec{b}$ be the vector connecting from a specific point on the plane (x_0, y_0, z_0) to our point (x_1, y_1, z_1).

$$\vec{n} =< a, b, c >$$

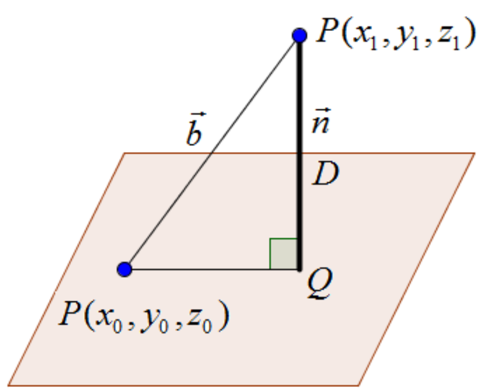
$$b =< x_1 - x_0, y_1 - y_0, z_1 - z_0 >$$

Here's the crux: the length D is the same as the positive value of the scalar projection of $\vec{b}$ onto $\vec{n}$. The formula for that is

$$\begin{aligned} D = |\text{comp}_{\vec{n}}\vec{b}| &= \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|} \\ &= \frac{| < a, b, c > \cdot < x_1 - x_0, y_1 - y_0, z_1 - z_0 > |}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Definition 5: Shortest Distance Between and Point and a Plane

et $P(x_1, y_1, z_1)$ be a point in three dimensional space (not on the plane), $P(x_0, y_0, z_0)$ be a point on the plane, and let $\vec{n} =< a, b, c >$ be a vector normal to the plane. Then the shortest distance from $P(x_1, y_1, z_1)$ to the plane is



$$D = \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 7

Find the distance between parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

We need two things: a point on the first plane, and a normal vector to the second plane.

1. Given any linear equation of a plane $ax+by+cz = d$, a normal vector is $\vec{n} =< a, b, c >$.

   So in our example we can use the normal vector $\vec{n} =< 5, 1, -1 >$.

2. Find any point (x, y, z) that works for $10x + 2y - z = 5$. I'll go with $(0, 0, -5/2)$.

3. Find D.

$$D = \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$
$$= \frac{|5(0) + 1(0) - 1(-5/2) - (5(0) + 1(0) - 1(-1))|}{\sqrt{5^2 + 1^2 + (-1)^2}}$$
$$= \frac{3/2}{\sqrt{27}}$$
$$= \frac{1}{2\sqrt{3}}$$