Math 232

Calculus III

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12.4 The Cross Product

Definition 1: The Cross Product

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then the cross product of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} =< a_2b_3 - a_3b_2, a_3b_1 - a_1, b_3, a_1b_2, a_2b_1 >$$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$$

This is hard to memorize. We have a better way of doing it. Before showing you I need to show you how to determine the determinates of a 2×2 and 3×3 matrices.

Definition 2: Determinate of a 2×2 Matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Definition 3: Determinate of a 3×3 Matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

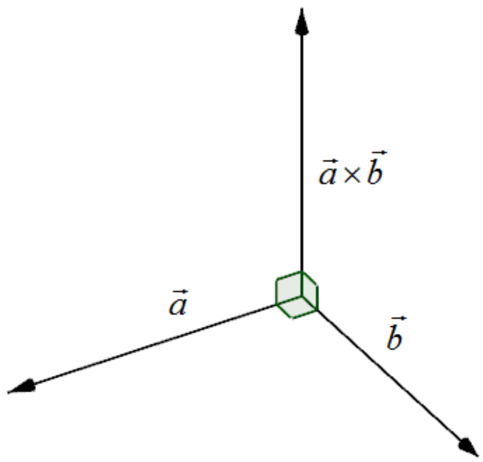
We can use the determinate of a 2×2 and 3×3 matrices to help with the formula for the cross product.

Definition 4: The Cross Product (Easier)

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

Please note the $(-)$ sign on the second determinate.

So what's the point. Seems like a lot of work for a formula. Suppose you have two vectors $\vec{a}$ and $\vec{b}$. You can find a unique plane that contains these two vectors. This will be discussed in a later section. Given any two vectors in $\mathbb{R}^3$, there is always a third vector that is perpendicular to both of them.



Note that

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{a} = 0$$

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{b} = 0$$

The direction of $\vec{a} \times \vec{b}$ uses the **Right Hand Rule**

Example 1

Find the cross product $\vec{a} \times \vec{b}$ of $\vec{a} =< 2, 3, 0 >$ and $\vec{b} =< 1, 0, 5 >$

For this problem we'll take it slow and do it step by step. Later on you'll be able to do this is one line.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} i - \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} j + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} k$$

1. $\begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} i = (3(5) - (0)(0))\, i = 15i$

2. $\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} i = (2(5) - (0)(1))\, j = -10j$

3. $\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} i = (2(0) - (3)(1))\, i = -3k$

Final Answer: $\vec{a} \times \vec{b} = 15i - 10j - 3k =< 15, -10, -3 >$

Just to make sure this is correct, we can check to see if $< 15, -10, -3 >$ is perpendicular to $\vec{a}$ and to $\vec{b}$.

$$< 15, -10, -3 > \cdot < 2, 3, 0 >= 30 - 30 - 0 = 0$$

$$< 15, -10, -3 > \cdot < 1, 0, 5 >= 15 - 0 - 15 = 0$$

Theorem 1

If θ is the angle between vectors $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin(\theta)$$

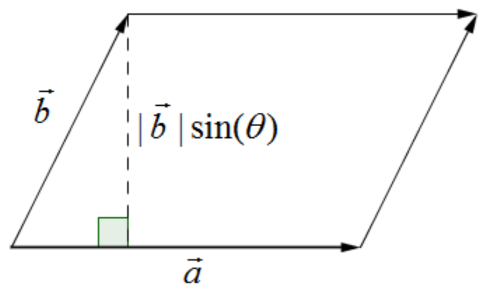
Parallel: If two vectors are parallel the angle between them is $\theta = 0$. And since $\sin(0) = 0$ it follows that

$$\vec{a} \times \vec{b} = 0 \text{ if } \vec{a} \text{ and } \vec{b} \text{ are parallel}$$

We also know that $\vec{a}$ and $\vec{b}$ are parallel if $\vec{a} = \lambda\vec{b}$ where λ is a scalar.

Definition 5: Applicaton of the Cross Product

Consider the following parallelogram formed by vectors $\vec{a}$ and $\vec{b}$



The area of this parallelogram is $|\vec{a}||\vec{b}|\sin(\theta)$. By the theorem above we know this is also $|\vec{a}\times\vec{b}|$. It follows that

$$|\vec{a}\times\vec{b}| = \text{ area of parallelogram}$$

Example 2

Find a vector perpendicular to the plane that passes through points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Three points form a plane. With those three points we can come up with two vectors. For example we can find the vector $\vec{PQ}$ and $\vec{PR}$.

$$\vec{PQ} =< -2-1, 5-4, -1-6 >=< -3, 1, -7 >$$

$$\vec{PR} =< 1-1, -1-4, 1-6 >=< 0, -5, -5 >$$

Since these two vectors lie on the same plane a vector perpedicular to the two vectors $\vec{PQ}$ and $\vec{PR}$ should also be perpendciular to the plane.

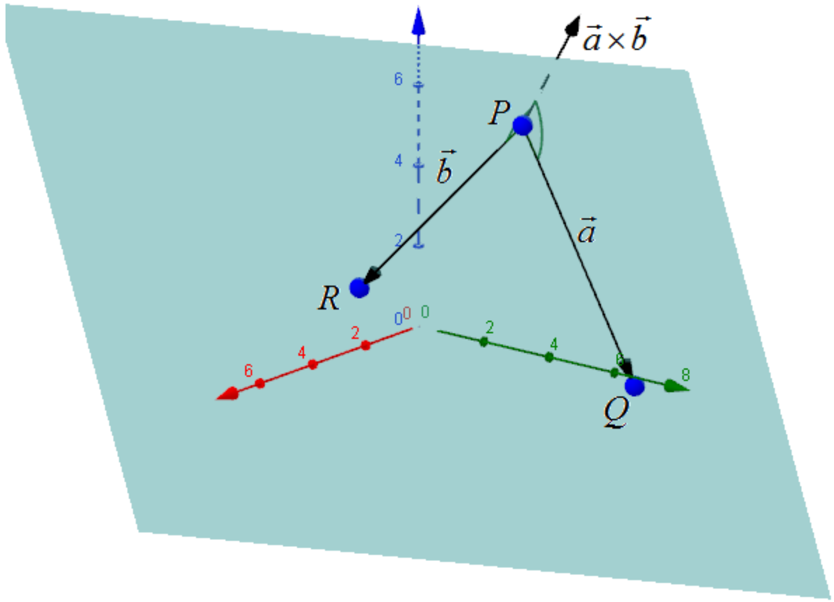
$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} i - \begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} j + \begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix} k$$

$$= (-5-35)i - (15-0)j + (15-0)k$$

$$= -40i - 15j + 15k$$

$$=< -40, -15, 15 >$$

Let's take a look at what's happening. The graph below will show the three points, the two vectors, and the plane they form.

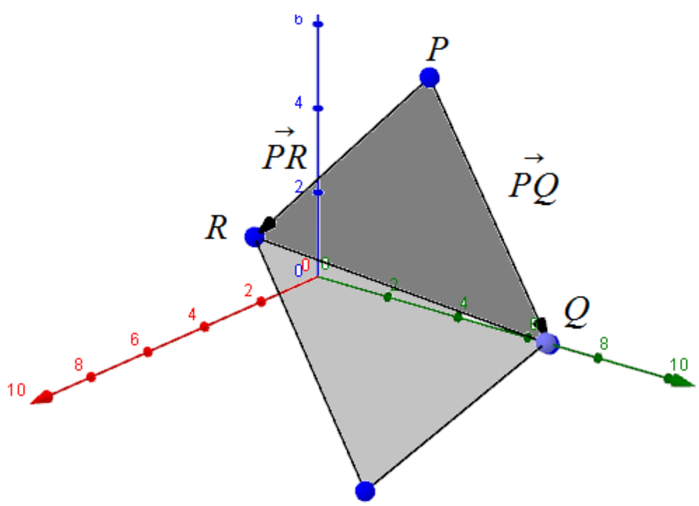


The perpendicular vector $\vec{a} \times \vec{b}$ is pointing behind the plane due to the Right Hand Rule. I also scaled the vector's length down so it would show up on the graph.

Example 3

Find the area of the triangle formed by the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Let's take a look at the triangle.



We are trying to find the area of the darker shaded region. We know from an application of the cross product that the area of the parallelogram formed by $\vec{PQ}$ and $\vec{PR}$ (both shaded regions) can be calculated by $|\vec{PQ} \times \vec{PR}|$.

$$\begin{aligned}\text{Area of Parallelogram } &= |\vec{PQ} \times \vec{PR}| = | < -40, -15, 15 > | \\ &= \sqrt{(-40)^2 + (-15)^2 + (15)^2} = \sqrt{2050}\end{aligned}$$

$$\begin{aligned}\text{Area of Triangle } &= \frac{1}{2}(\text{Area of Paralleogram}) \\ &= \frac{1}{2}\sqrt{2050} \\ &\approx 22.6\end{aligned}$$

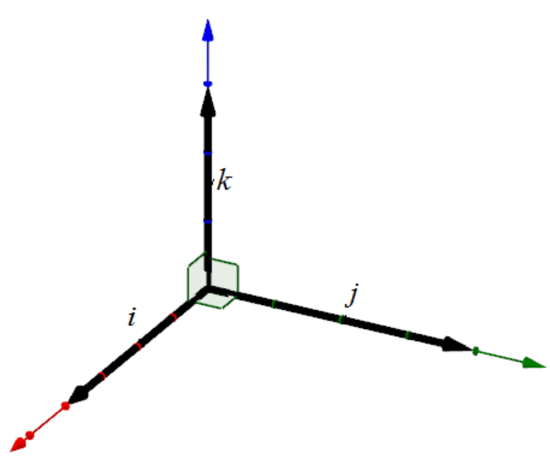
Returning back to the standard basis vectors i, j, and k, let's talk about their cross products.

Definition 6

$$i \times j = k \qquad j \times k = i \qquad k \times i = j$$

$$j \times i = -k \qquad k \times j = -i \qquad i \times k = -j$$

This should make sense since i represents the x-axis, j represents the y-axis, and k represents the z-axis. So the cross product of the x and y axis should be the z axis. The reason the (-) signs are showing up is due to the Right Hand Rule.



Definition 7: Properties of Cross Prodcuts

1. $\vec{a} \times \vec{b} = -\vec{a} \times \vec{b}$

2. $(r\vec{a}) \times \vec{b} = r(\vec{a} \times \vec{b}) = \vec{a} \times (r\vec{b})$

3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

4. $(a \vec{+} b) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$