Math 232

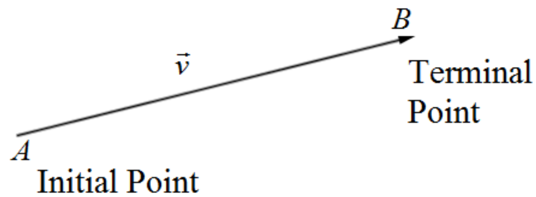
Calculus III

Brian Veitch • Fall 2015 • Northern Illinois University

12.2 Vectors

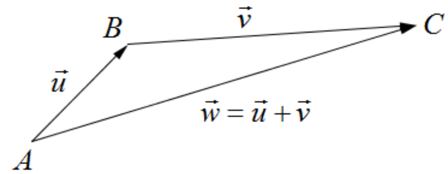
Definition 1: Vector

A vector is a quantity that has both direction and magnitude (length). We write $\vec{v} = \vec{AB}$



Definition 2: Vector Addition

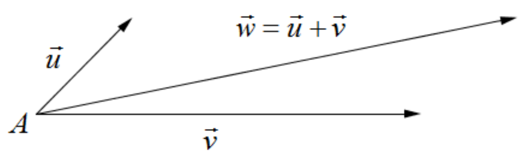
Geometrically: Let $\vec{u} = \vec{AB}$ and $\vec{v} = \vec{BC}$



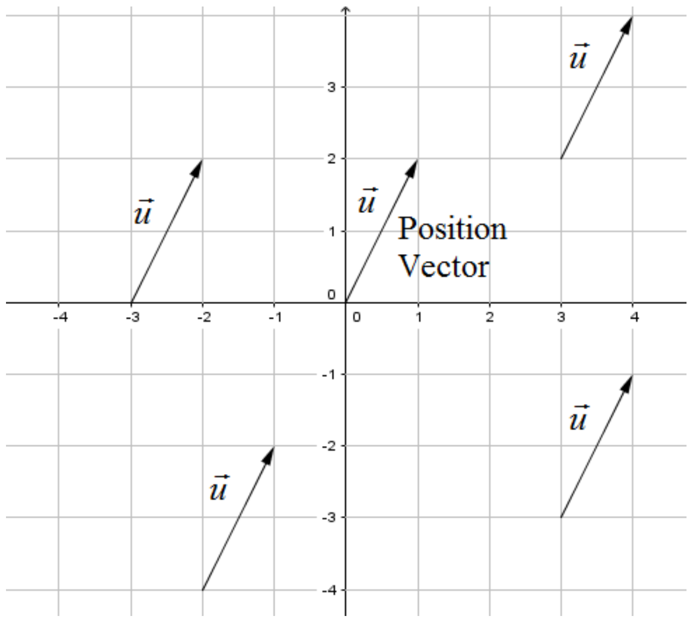
$$\vec{AB} + \vec{BC} = \vec{AC}$$

where $\vec{AC} = \vec{u} + \vec{v}$. This is called the **Head to Tail** method of Vector Addition.

Vectors Sharing the Same Initial Point

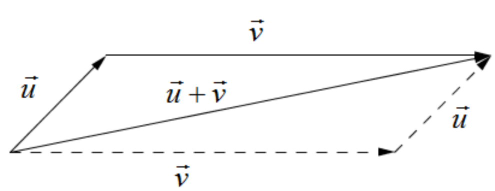


To show why $\vec{u}+\vec{v}$ is the vector shown above we need to first understand a vector a little more. Because a vector is defined to be simply length and direction, any vector with those two properties are considered equivalent. Each vector in the graph below are considered the same vector $\vec{u}$.



Position Vector: The position vector is always the vector that has its initial point at the origin $(0,0)$.

Let's get back to vector addition. If both vectors share the same initial point you can move one of the vectors (making sure to keep the same direction and length) so that the tail of $\vec{v}$ starts at the head of $\vec{u}$. Then using the originally defined vector addition from above we get



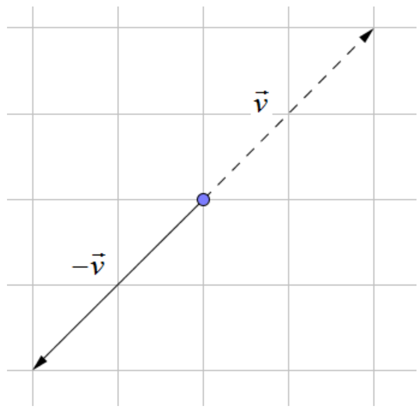
You can also see that it forms a parallelogram. This is known as the **Parallelogram Law**. This also shows that

$$\vec{u}+\vec{v}=\vec{v}+\vec{u}$$

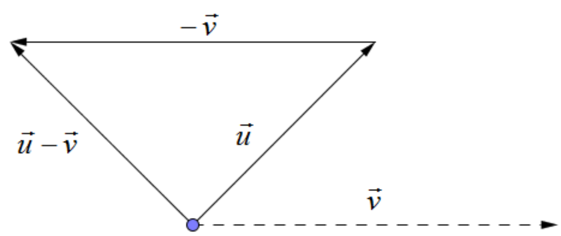
Definition 3: Vector Subtraction

Let $\vec{u}$ and $\vec{v}$ be vectors. Then $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

What's $-\vec{v}$? You rotate the vector $\vec{u}$ 180 degrees so that it's pointing in the opposition direction.



Again given vectors $\vec{u}$ and $\vec{v}$, then $\vec{u} - \vec{v}$ is shown below

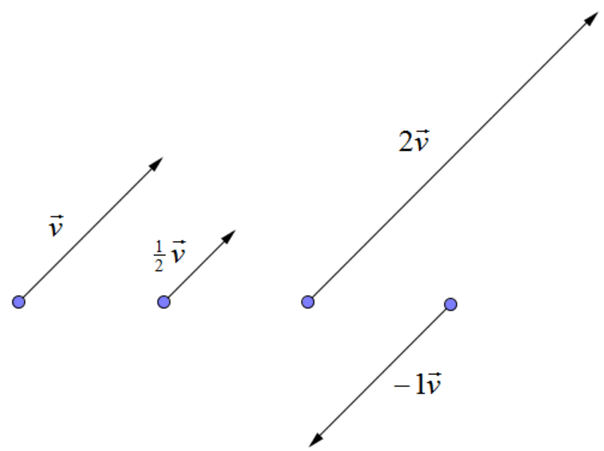


Definition 4: Scalar Multiplication

Multiplying a vector $\vec{v}$ by a scalar c will stretch or compress the vector.

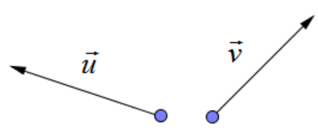
If $c > 0$ the vector $c\vec{v}$ keeps the same direction.

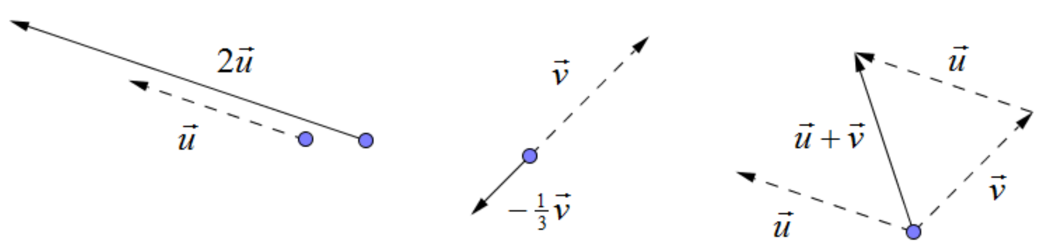
If $c < 0$ the vector $c\vec{v}$ points in the opposite direction.

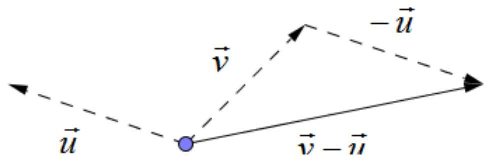


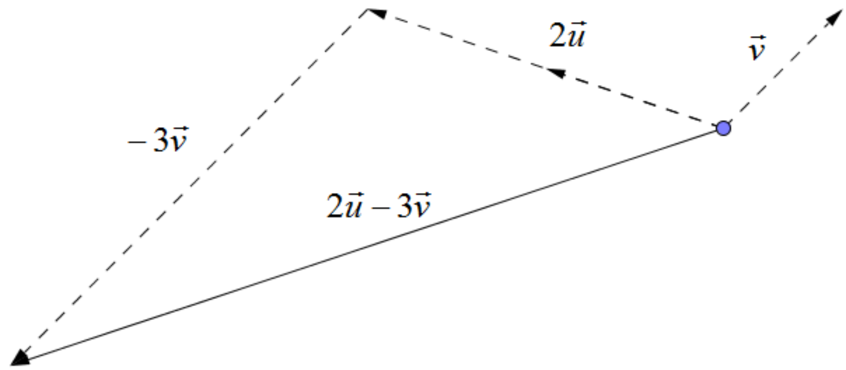
Example 1

If vectors $\vec{u}$ and $\vec{v}$ are defined below, find the vectors $2\vec{u}$, $-\frac{1}{3}\vec{v}$, $\vec{u}+\vec{v}$, $\vec{v}-\vec{u}$, and $2\vec{u}-3\vec{v}$



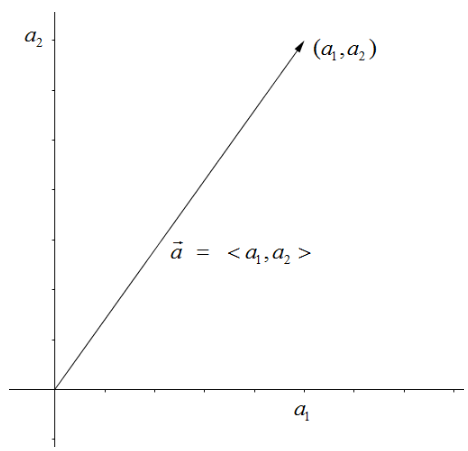


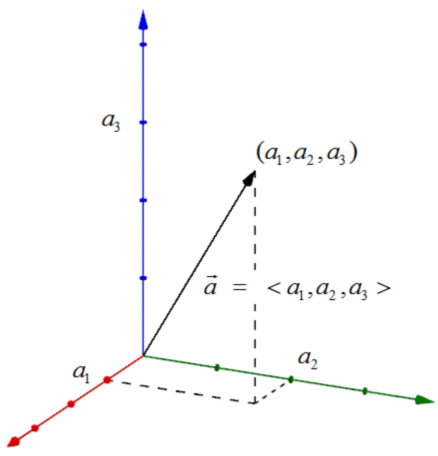




Definition 5: Vectors Componenets

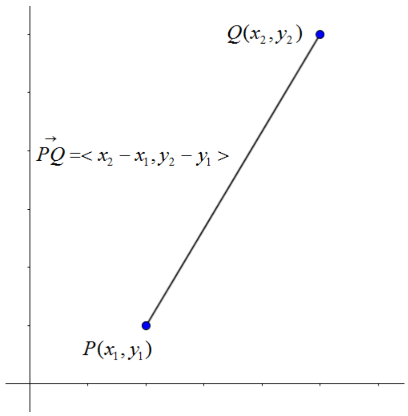
Let $\vec{a}$ be a vector defined by $\vec{a} =< a_1, a_2 >$ or $\vec{a} =< a_1, a_2, a_3 >$. a_1, a_2 and a_3 are called the components of vector $\vec{a}$.

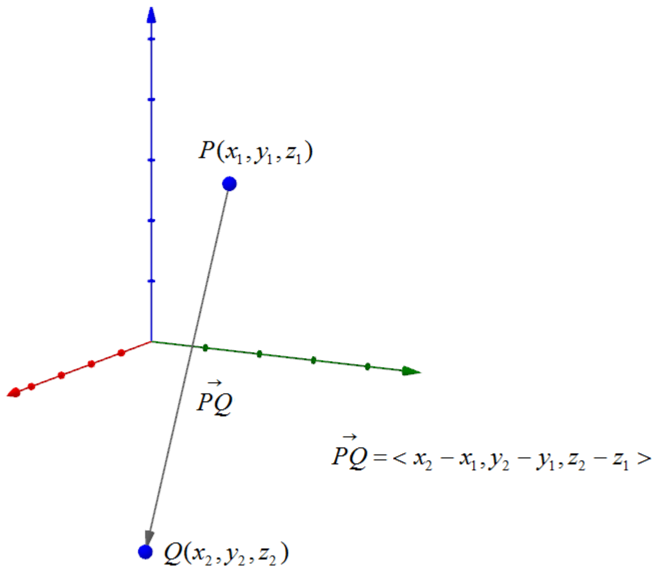




The components are the displacement from the initial point to its terminal.

Definition 6: Creating a Vector from Two Points





Definition 7: Vector Magnitude (length)

Let $\vec{a} =< a_1, a_2 >$, then the magnitude is $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$

Let $\vec{a} =< a_1, a_2, a_3 >$, then the magnitude is $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Definition 8: Vector Addition/Subtraction, Scalar Multiplication

For 2D: Let $\vec{a} =< a_1, a_2 >$, $\vec{b} =< b_1, b_2 >$, and c be a scalar, then

$$\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2 >$$

$$\vec{a} - \vec{b} =< a_1 - b_1, a_2 - b_2 >$$

$$c\vec{a} =< ca_1, ca_2 >$$

For 3D: Let $\vec{a} =< a_1, a_2, a_3 >$, $\vec{b} =< b_1, b_2, b_3 >$, and c be a scalar, then

$$\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 >$$

$$\vec{a} - \vec{b} =< a_1 - b_1, a_2 - b_2, a_3 - b_3 >$$

$$c\vec{a} =< ca_1, ca_2, ca_3 >$$

Example 2

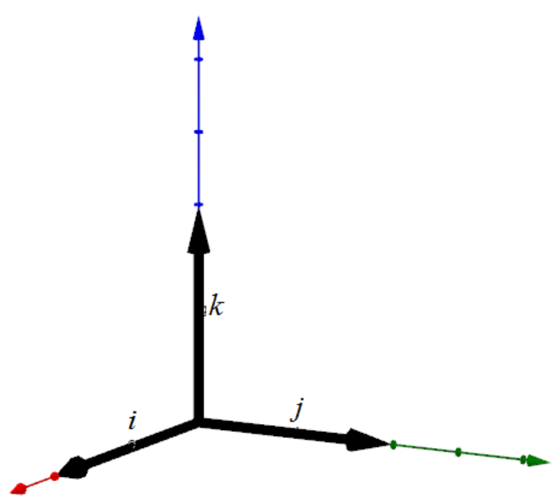
Let $\vec{a} =< 4, 0, 3 >$ and $\vec{b} =< -2, 1, 5 >$. Find $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $3\vec{b}$, $|\vec{a}|$, $|\vec{b}-\vec{a}|$, and $2\vec{a}+4\vec{b}$

1. $\vec{a}+\vec{b} =< 4, 0, 3 > + < -2, 1, 5 >=< 2, 1, 7 >$
2. $\vec{a}-\vec{b} =< 4, 0, 3 > - < -2, 1, 5 >=< 6, -1, -2 >$
3. $3\vec{b} = 3 < -2, 1, 5 >=< -6, -3, 15 >$
4. $|\vec{a}| = \sqrt{(4)^2 + (0)^2 + (3)^2} = \sqrt{25} = 5$
5. $|\vec{b}-\vec{a}| = | < -6, 1, 2 > | = \sqrt{(-6)^2 + (1)^2 + (2)^2} = \sqrt{36 + 1 + 4} = \sqrt{41}$
6. $2\vec{a} + 5\vec{b} = 2 < 4, 0, 3 > +5 < -2, 1, 5 >=< 8, 0, 6 > + < -10, 5, 25 >=< -2, 5, 31 >$

Note that all arithmetic properties are true. For example, $\vec{a}+\vec{b} = \vec{b}+\vec{a}$

Definition 9: Standard Basis Vectors (Unit Vectors)

$$i =< 1, 0, 0 >, \quad j =< 0, 1, 0 >, \quad k =< 0, 0, 1 >$$



This means the vector $\vec{a} =< a_1, a_2, a_3 >$ can be written as $\vec{a} = a_1 i + a_2 j + a_3 k$

For example, $\vec{a} =< -1, 4, 6 >= -i + 4j + 6k$ or $\vec{b} =< 4, 2 >= 4i + 2j$.

$$\vec{a}+\vec{b} = (-i + 4j + 6k) + (4i + 2j + 0k) = 3i + 6j + 6k =< 3, 6, 6 >$$

Definition 10: Unit Vector

A unit vector is a vector with length 1. If $\vec{a}$ is any vector, then

$$\frac{\vec{a}}{|\vec{a}|} \text{ is a unit vector}$$

To find a vector with the direction of $\vec{a}$ with length L

$$\vec{v} = \frac{L}{|\vec{a}|}\vec{a}$$

Example 3

Let $\vec{a} =< 2, -1, -2 >$. Find the unit vector with the direction of $\vec{a}$.

1. First, find $|\vec{a}|$.

$$|\vec{a}| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

2. To create a unit vector, divide $\vec{a}$ by 3.

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3} < 2, -1, -2 >=< \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} >$$