Math 232

Calculus III

Brian Veitch • Fall 2015 • Northern Illinois University

10.1 - Curves Defined By Parametric Equations

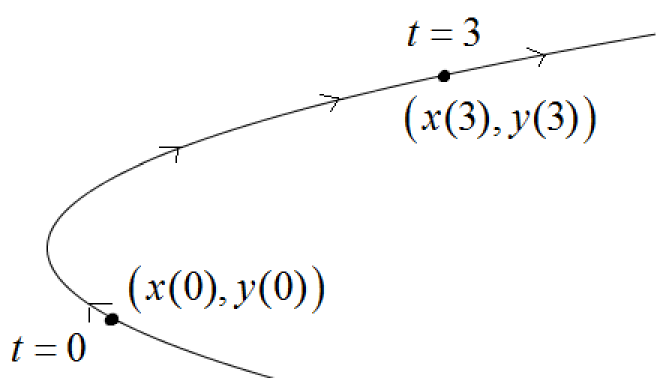
Definition 1: Parametric Equations

We define a point $P(x, y)$ in the xy-plane. Suppose x and y are both given as continuous functions of a variable t (our parameter). In this case, we could write

$$x = x(t) \text{ or } x = f(t)$$

$$y = y(t) \text{ or } y = g(t)$$

We say that $P(f(t), g(t))$ is the parametric representation of $P(x, y)$.



Note: y does not depend on x. Both are dependent on the parameter t.

Example 1

Let $x = a\cos(t)$ and $y = \sin(t)$.

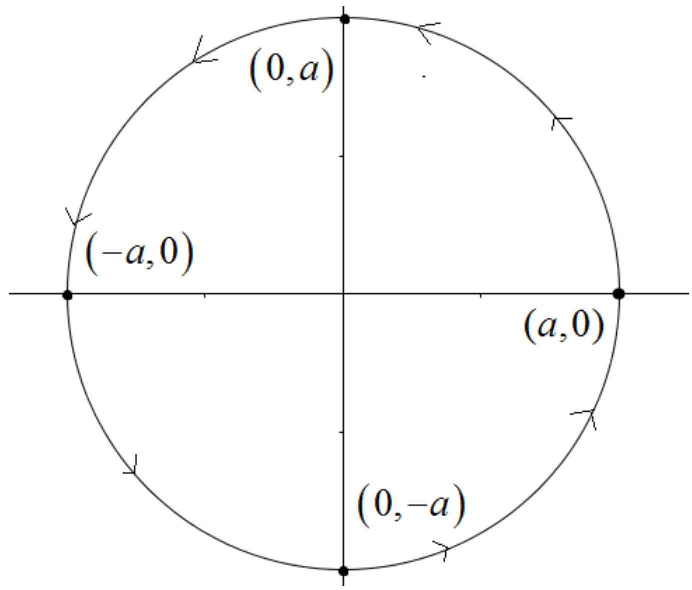
Before getting into sketching the curve, note that $x^2 + y^2 = a^2\cos^2(t) + a^2\sin^2(t) = a^2$. We know $x^2 + y^2 = a^2$ is a circle centered at $(0, 0)$ with radius a.

Steps 1: Sketching a Paremetric Curve

1. Make a t-table with columns for t, x, and y.
2. Choose t values (from the domain) like $t = -1, 0, 1, 2, ...$ if the equations are rational or polynomial-ish.
3. Choose t values (from the domain) like $t = 0, \pi/4, \pi/2, \pi$, etc., if the equations are trigonometric.
4. Plot and connect the points (note the direction with arrows)

For $x = a\cos(t)$ and $y = a\sin(t)$ the table would look like this:

t	x	y
0	a	0
$\pi/2$	0	a
π	$-a$	0
$3\pi/2$	0	$-a$
2π	a	0



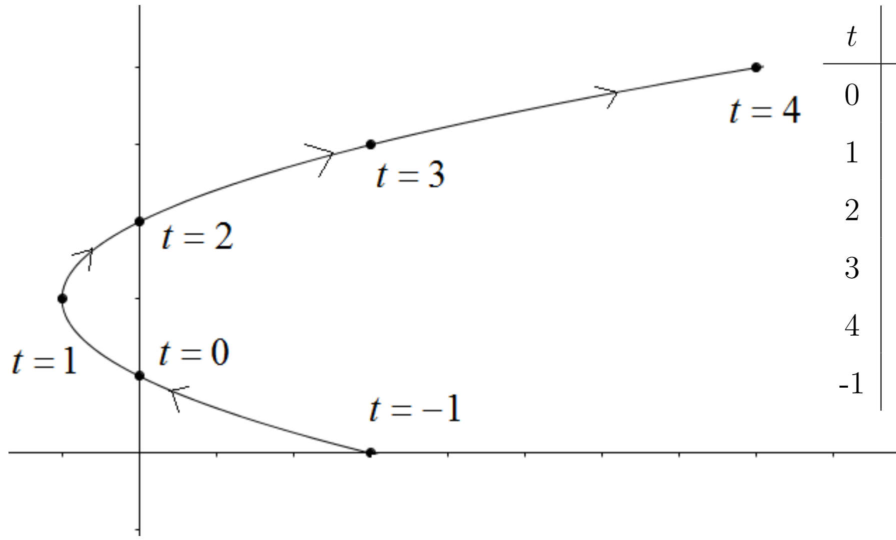
Note: After $t = 2\pi$ the curve repeats itself. This is an issue later on when we want to find the arc length.

Example 2

Sketch the curve defined by $x = t^2 - 2t$ and $y = t + 1$ from $-1 \le t \le 4$

Following the steps listed above let's start with a table.

Plot the points, connect the dots, and mark the directions we get the following graph:



t	x	y
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5
-1	3	0

- In this example $(x(-1), y(-1))$ is called the **Initial Point**
- In this example $(x(4), y(4))$ is called the **Terminal Point**

Sometimes it's possible to write a parametric function $x(t)$ and $y(t)$ into a function of y and x by removing the parameter t.

1. Solve one of the parametric equations for t (if possible)

$$y = t + 1 \Rightarrow t = y - 1$$

2. Plug $t = y - 1$ into the other parametric function.

$$x = t^2 - 2t \Rightarrow x = (y-1)^2 - 2(y-1)$$

$$x = y^2 - 2y + 1 - 2y + 2$$

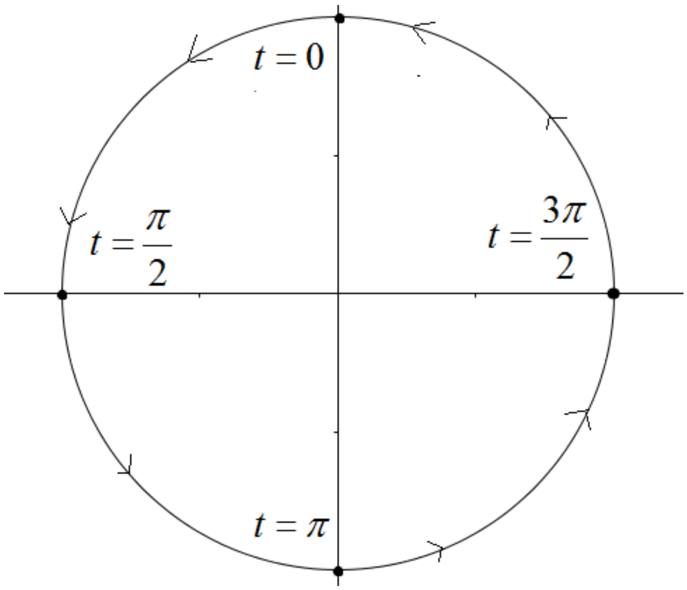
$$x = y^2 - 4y - 3$$

If you sketch $x = y^2 - 4y - 3$ you will get the same graph as above.

Example 3

Sketch the curve given by $x = \sin(3t)$, $y = \cos(3t)$, $0 \le t \le 2\pi$

Like the first example notice that $x^2 + y^2 = \sin^2(3t) + \cos^2(3t) = 1$. This will be a circle centered at $(0,0)$ with radius 1. Let's look at the t-table.

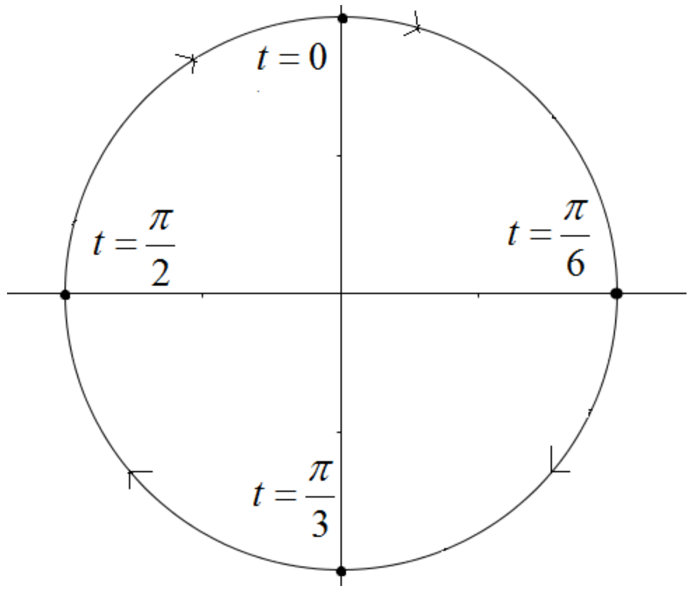


t	0	1
$\pi/2$	-1	0
π	0	-1
$3\pi/2$	1	0
2π	0	1

There is one big problem that you may or may not have noticed. The direction is actually wrong. Let's discuss why. This looks like the parametric equations make one full revolution on $[0, 2\pi]$. But $\sin(3t)$ does not have a period of $t = 2\pi$. From trig you would have learned that the period is actually $3t = 2\pi \Rightarrow t = 2\pi/3$. This means in the domain $[0, 2\pi]$ the circle completes 3 revolutions at $t = 2\pi/3$, $4\pi/3$, and $6\pi/3 = 2\pi$. Let's take a look at the smaller domain of $[0, 2\pi/3]$

t	0	1
0	0	1
$\pi/6$	1	0
$\pi/3$	0	-1
$\pi/2$	-1	0
$2\pi/3$	0	1

Now we can see a full revolution of the circle **AND** the correct direction of the movement.



Example 4

Sketch $x = 2\sin(t)$ and $y = \cos(t)$

Let's try to rewrite this into a function of just x and y.

$$x^2 + y^2 = 4\sin^2 t + \cos^2 t \neq 0$$

We can't do the same trick as before because of the 2 in $2\sin(t)$. Instead try

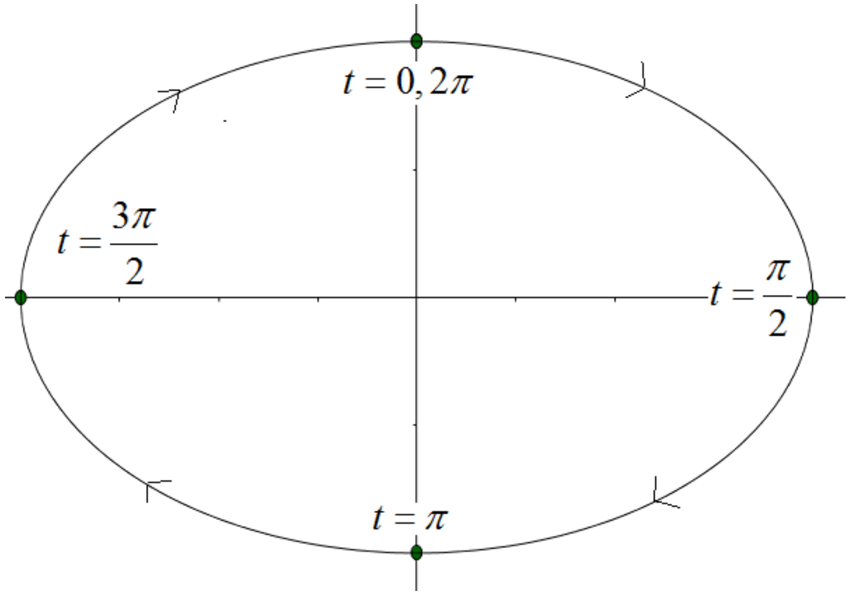
$$\frac{1}{4}x^2 + y^2 = \frac{1}{4} \cdot 4\sin^2(t) + \cos^2(t) = 1$$

$$\frac{1}{4}x^2 + y^2 = 1$$

So this graph will look like an ellipse. We'll talk more about ellipses later. Let's make our table. Because the parameter in the trig functions is just t we can look at the domain $[0, 2\pi]$.

t	x	y
0	0	1
$\pi/2$	2	0
π	0	-1
$3\pi/2$	-2	0
2π	0	1

Plotting the points, connecting the dots, and marking the direction we get the following graph:



Example 5

Sketch $x = 5 + 2\cos(\pi t)$ and $y = 3 + 2\sin(\pi t)$, $1 \le t \le 2$

In order to remove the parameter t, solve both equations for the trig part of it.

$$x = 5 + 2\cos(\pi t) \Rightarrow \cos(\pi t) = \frac{x-5}{2}$$

$$y = 3 + 2\sin(\pi t) \Rightarrow \sin(\pi t) = \frac{y-3}{2}$$

Doing our trick we get

$$\cos^2(\pi) + \sin^2(\pi t) = \left(\frac{x-5}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$$

This gives us

$$(x-5)^2 + (y-3)^2 = 4 \text{ a circle centered at (5,3) with radius 2}$$

To make the table I'll break the domain $1 \le t \le 2$ into four parts.

t	x	y
1	3	3
5/4	3.6	1.59
5/2	5	5
7/4	6.41	1.59
2	7	3

Plot the points, connect the dots, and mark the direction and we get