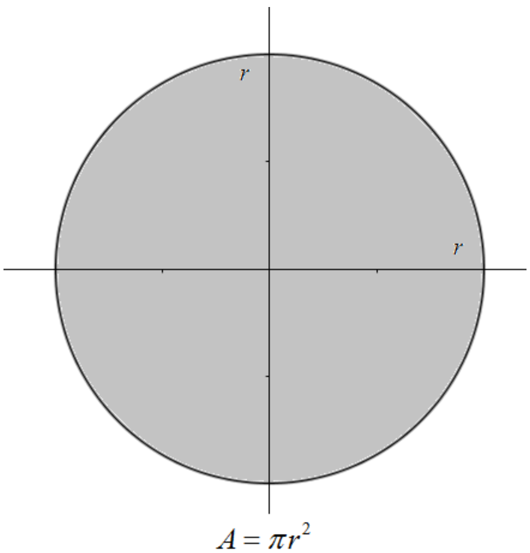
Math 232

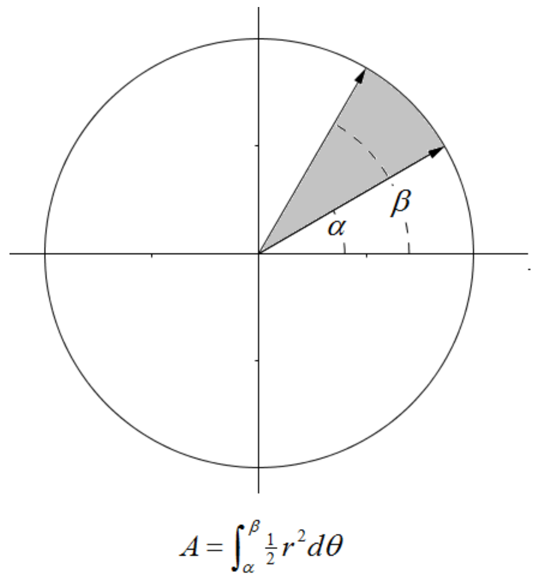
Calculus III

Brian Veitch • Fall 2015 • Northern Illinois University

10.4 Areas and Lengths in Polar

Consider the area of a circle with radius r and part of that circle





Definition 1: Area of a Polar Region

The area for finding area enclosed under a polar curve is

$$A = \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$

Trig Identities you will need are

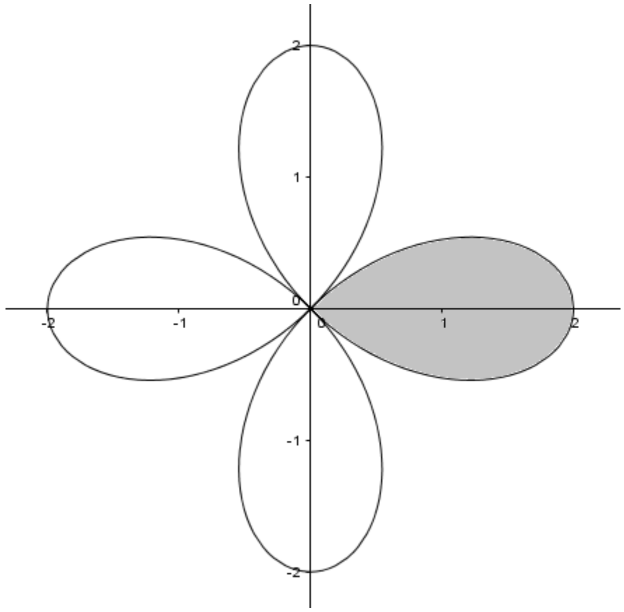
$$\sin^2(\theta) = \frac{1}{2}\left(1 - \cos(2\theta)\right)$$

$$\cos^2(\theta) = \frac{1}{2}\left(1 + \cos(2\theta)\right)$$

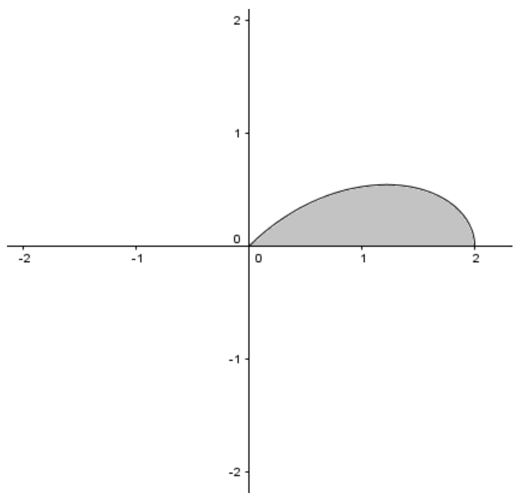
Example 1

Find the area of one loop of $r = 2\cos(2\theta)$.

Using the techniques from section 10.3, the graph of $r = 2\cos(2\theta)$ is



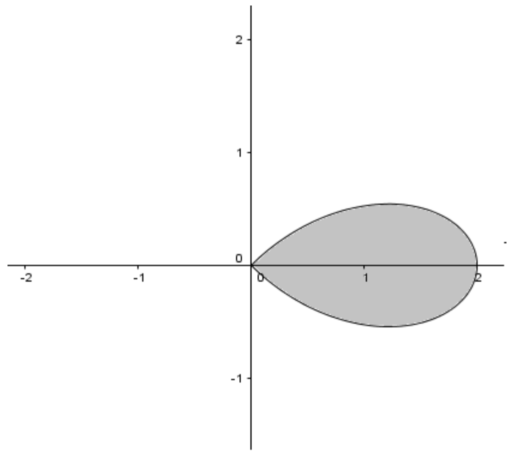
So how to we find the two values for θ that will enclose the shaded region from above. Remember you should be marking the points and direction as you connect dots. If you plot points in $[0, \pi/4]$ you have this part of the curve:



This gives us the top part of the region. You could stop here and evaluate

$$A = 2 \cdot \int_0^{\pi/4} \frac{1}{2} \left(2\cos(2\theta)\right) \, d\theta$$

since the top part and bottom part are equal. Or you can figure out how to graph the bottom part of the loop. Playing around with values of θ and you get the bottom part on the interval $[-\pi/4, 0]$.

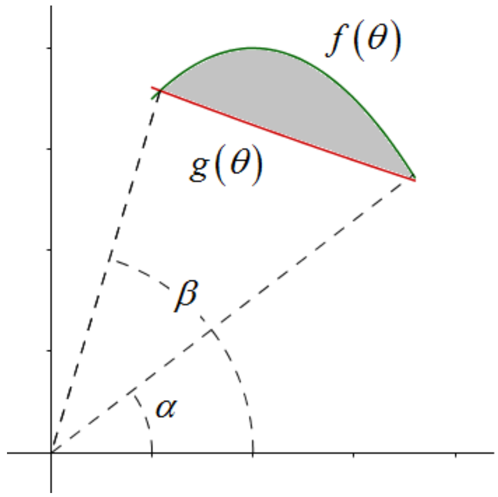


$$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

$$\begin{aligned}
A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{2} \left(2\cos(2\theta)\right)^2 \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} 2\cos^2(2\theta) \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} 2 \cdot \frac{1}{2} \left(1 + \cos(4\theta)\right) \, d\theta \\
&= \int_{-\pi/4}^{\pi/4} 1 + \cos(4\theta) \, d\theta \\
&= \theta + \frac{1}{4}\sin(4\theta) \Big|_{-\pi/4}^{\pi/4} \\
&= \left[\frac{\pi}{4} + \frac{1}{4}\sin(\pi)\right] - \left[-\frac{\pi}{4} + \frac{1}{4}\sin(-\pi)\right] \\
&= \pi/2
\end{aligned}$$

Definition 2: Area Between Polar Curves

Consider the following set of curves below.

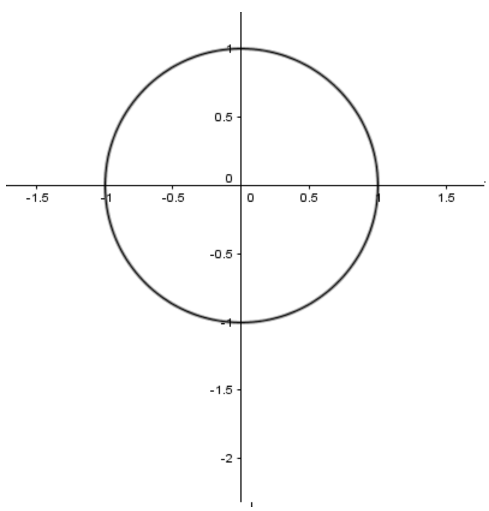


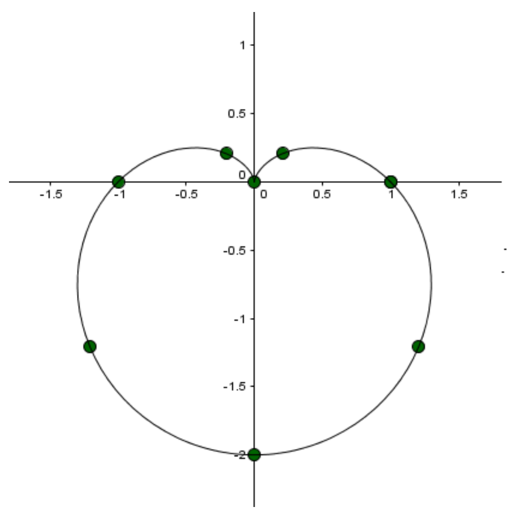
$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 - \frac{1}{2}\left(g(\theta)\right)^2 \ d\theta$$

Example 2

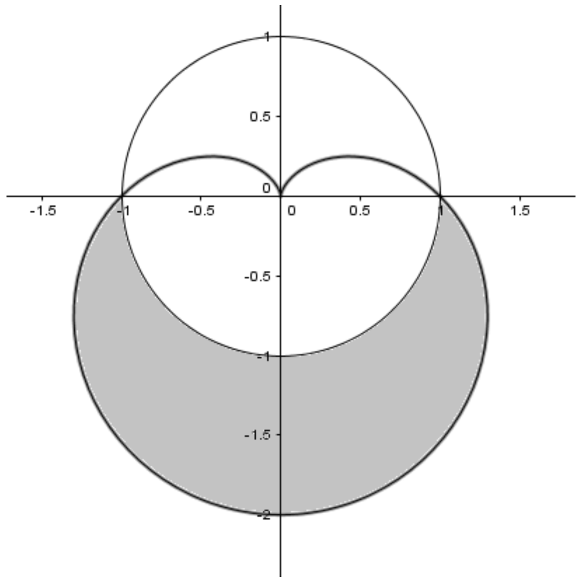
Find the area outside $r = 1$ but inside $r = 1 - \sin(\theta)$.

Here's the graph of each function:





Placing them on on the same graph we get

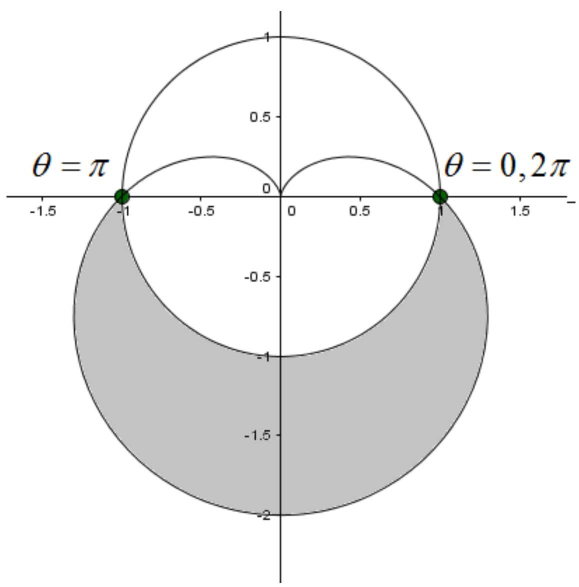


First thing, let's figure out where the two curves intersect.

$$1 = 1 - \sin(\theta)$$

$$\sin(\theta) = 0$$

$$\theta = 0, \pi, 2\pi$$



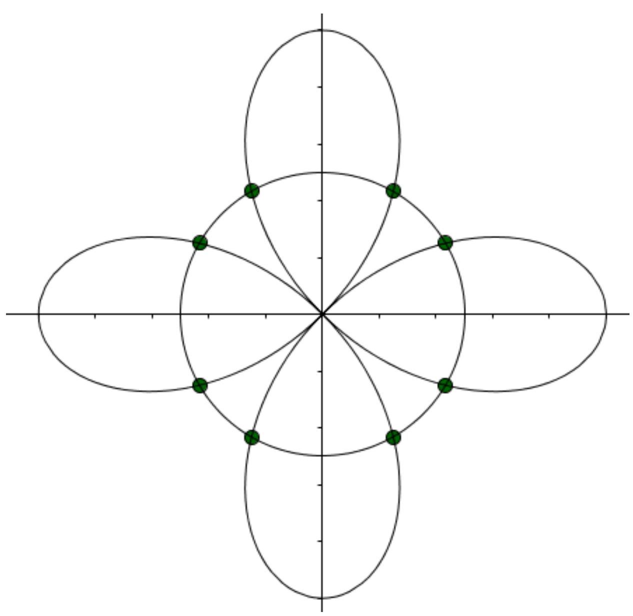
If you follow the interval $[0, \pi]$ you trace along the top portion of $r = 1 - \sin(\theta)$ which would not enclose the area we want. If you follow the interval $[\pi, 2\pi]$ we get the bottom portion of $r = 1 - \sin(\theta)$ that we want.

$$\begin{aligned}
A &= \int_{\pi}^{2\pi} \text{Outside - Inside } \, d\theta \\
&= \int_{\pi}^{2\pi} \frac{1}{2}\left(1-\sin(\theta)\right)^2 - \frac{1}{2}(1)^2 \, d\theta \\
&= \int_{\pi}^{2\pi} \frac{1}{2}\left(1-2\sin(\theta)+\sin^2(\theta)\right) - \frac{1}{2} \, d\theta \\
&= \int_{\pi}^{2\pi} -\sin(\theta) + \frac{1}{2}\sin^2(\theta) \, d\theta \\
&= \int_{\pi}^{2\pi} -\sin(\theta) + \frac{1}{4} - \frac{1}{4}\cos(2\theta) \, d\theta \\
&= \cos(\theta) + \frac{1}{4}\theta - \frac{1}{8}\sin(2\theta)\Big|_{\pi}^{2\pi} \\
&= \left[\cos(2\pi) + \frac{1}{4}(2\pi) - \frac{1}{8}\sin(4\pi)\right] - \left[\cos(\pi) + \frac{1}{4}\pi - \frac{1}{8}\sin(2\pi)\right] \\
&= 1 + \pi/2 - 0 + 1 - \pi/4 - 0 \\
&= 2 + \pi/4
\end{aligned}$$

Example 3

Consider the two polar curves $r = \cos(2\theta)$ and $r = 1/2$. How many intersection points are there?

Here's the graph we're dealing with



From the graph you can see there are 8 intersection points. They occur when $\cos(2\theta) = \dfrac{1}{2}$.

$$\cos(2\theta) = \frac{1}{2}$$

$$\Rightarrow 2\theta = \pi/3, 5\pi/3$$

$$\Rightarrow \theta = \pi/6, 5\pi/6$$

This only gives us two of the solutions. Here's the problem. We need to now ignore the graphs and stick just to the trig equations. The period on $\cos(2\theta)$ is π (NOT 2π). The period of $r = \dfrac{1}{2}$ is 2π. This means we actually make two revolutions of $\cos(2\theta)$. by the time $r = 1/2$ finishes its revolution. This actually means

$$2\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi, 3$$

$$\Rightarrow \theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$$

Wait, this is only four solutions. We need 8. The rest of the solutions comes from the fact that the polar equation $r = 1/2$ can also be represented by $r = -1/2$. We need to go through solving $\cos(2\theta) = -1/2$ with the same argument as above.

$$2\theta = 2\pi/3, 4\pi/3, 8\pi/3, 10\pi/3$$

$$\theta = 2\pi/6, 4\pi/6, 8\pi/6, 10\pi/6$$

$$\theta = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$$

.

Arc Length

Definition 3: Arc Length with Polar Curves

Let $r = f(\theta)$. The length of r on $\alpha \leq \theta \leq \beta$ is

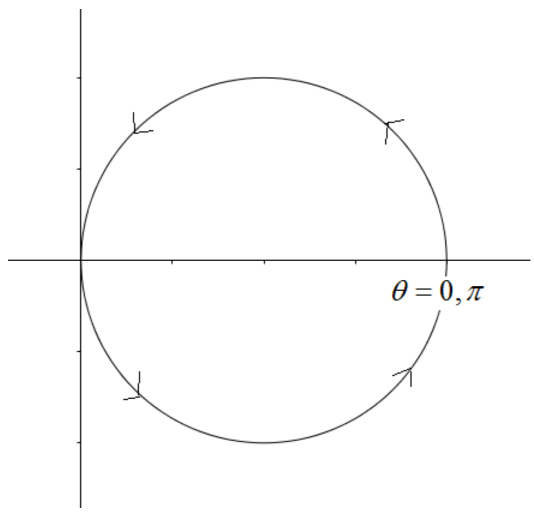
$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

Note: You have to make sure you don't traverse the curve more than once. For example, going around a circle twice.

Example 4

Find the arc length of $r = 2\cos(\theta)$.

The graph looks like



Notice that it makes a full revolution on the interval $[0, \pi]$. And $\frac{dr}{d\theta} = -2\sin(\theta)$. The arc length is then

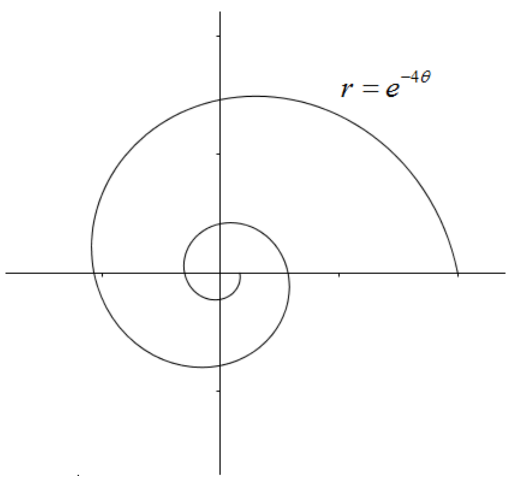
$$\begin{aligned} L &= \int_0^{\pi} \sqrt{(-2\sin(\theta))^2 + (2\cos(\theta))^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{4\sin^2(\theta) + 4\cos^2(\theta)}\, d\theta \\ &= \int_0^{\pi} \sqrt{4}\, d\theta \\ &= \int_0^{\pi} 2\, d\theta \\ &= 2\theta|_0^{\pi} \\ &= 2\pi \end{aligned}$$

Which makes sense as this is a circle with radius 1.

Example 5

Find the arc length of $r = e^{-4\theta}$, $0 \leq \theta \leq 4\pi$

The graph looks something like



$$\frac{dr}{d\theta} = -4e^{-4\theta}$$

$$\begin{aligned}
L &= \int_0^{4\pi} \sqrt{\left(-4e^{-4\theta}\right)^2 + \left(e^{-4\theta}\right)^2}\, d\theta \\
&= \int_0^{4\pi} \sqrt{16e^{-8\theta} + e^{-8\theta}}\, d\theta \\
&= \int_0^{4\pi} \sqrt{17e^{-8\theta}}\, d\theta \\
&= \int_0^{4\pi} \sqrt{17}e^{-4\theta}\, d\theta \\
&= -\frac{\sqrt{17}}{4}e^{-4\theta}\Big|_0^{4\pi} \\
&= -\frac{\sqrt{17}}{4}e^{16\pi} + \frac{\sqrt{17}}{4} \\
&\approx 1.03
\end{aligned}$$