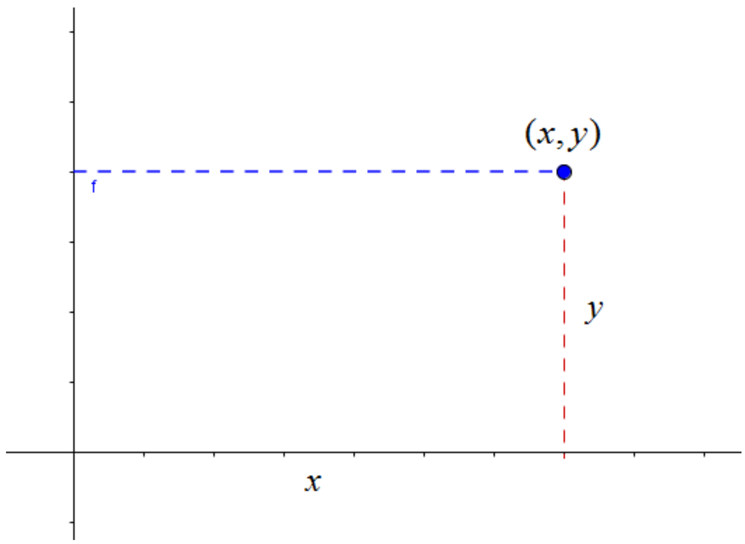
Math 232

Calculus III

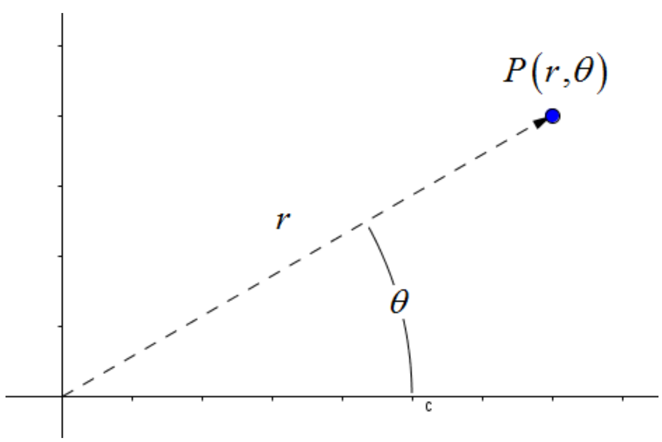
Brian Veitch • Fall 2015 • Northern Illinois University

10.3 Polar Coordinates

Consider the rectangular coordinate system.



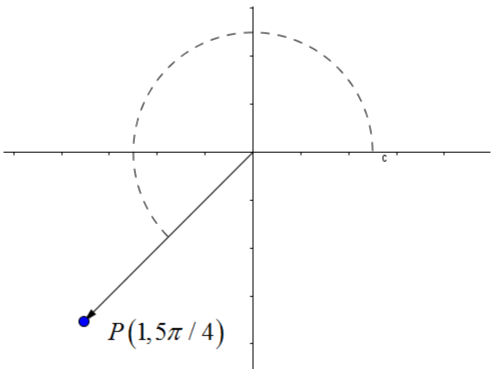
We want to find another way to get to the point (x, y). One way to do this is to use an angle θ and a distance r. It will look like this



Example 1

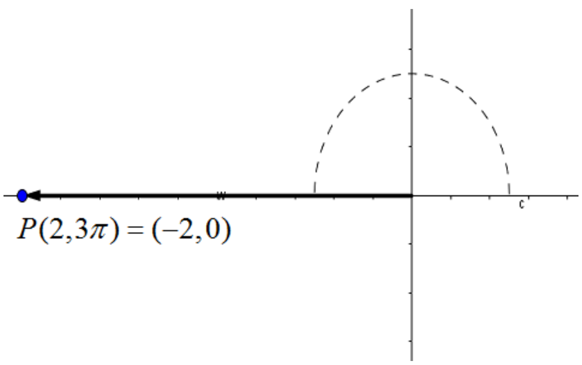
Plot the following polar points $(1, 5\pi/4)$, $(2, 3\pi)$, $(2, -2\pi/3)$, and $(-3, 5\pi/4)$ on the xy plane

1. $P(1, 5\pi/4)$



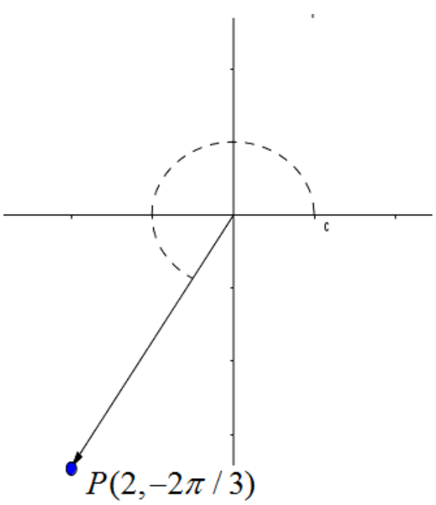
Can also use the points $P(1, -3\pi/4)$ or $P(-1, \pi/4)$

2. $P(2, 3\pi)$



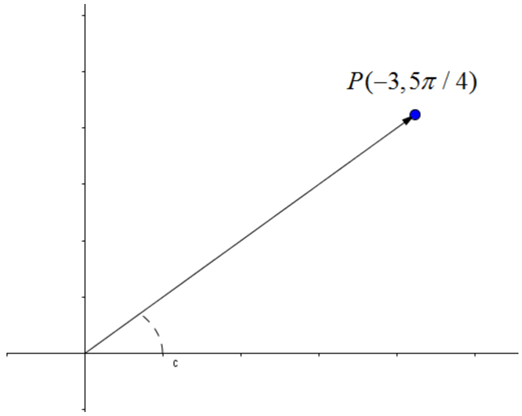
Can also use the points $P(2, \pi)$ or $P(-2, 0)$

3. $P(2, -2\pi/3)$



Can also use the points $P(-2, \pi/3)$ or $P(2, 4\pi/3)$

4. $P(-3, 5\pi/4)$

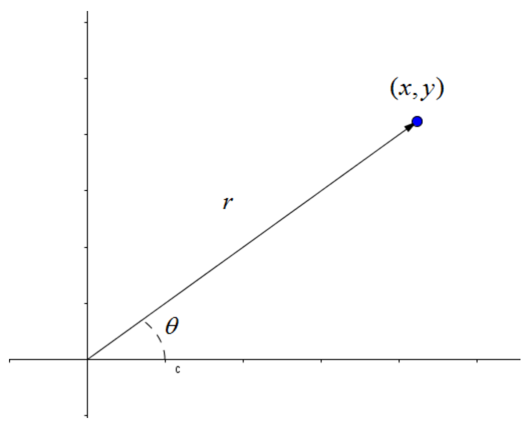


Can also use the points $P(-3, -3\pi/4)$ or $P(3, \pi/4)$

One of the biggest things to note from the previous problem is polar coordinates are NOT unique. I listed two additional polar coordinates for each of the problems above that get you to the same point.

Formula 1: Fundemental Formula for Polar Coordinates

Given the following following point (x, y)



We have the following relationships

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta)$$

$$r^2 = x^2 + y^2 \text{ and } \tan(\theta) = \frac{y}{x}$$

Example 2

Convert each of the following points to the given coordinate system.

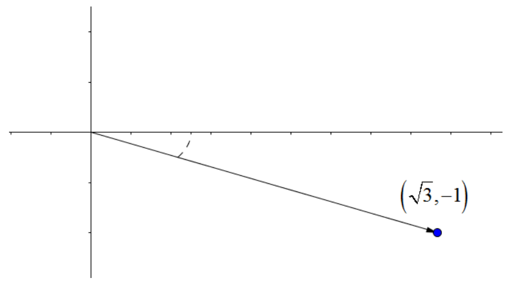
1. $(2, -\pi/6)$ into rectangular
2. $(-1, 1)$ into polar coordinates

1. $(2, -\pi/6)$

$$x = r\cos(\theta) = 2\cos(-\pi/6) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$y = r\sin(\theta) = 2\sin(-\pi/6) = 2\left(\frac{-1}{2}\right) = -1$$

The point in rectangular coordinates: $(\sqrt{3}, -1)$

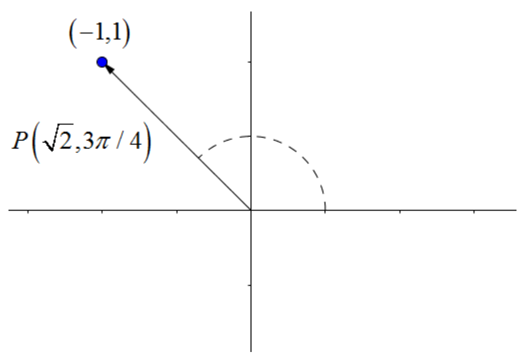


2. $(-1, 1)$ into Polar

$$r^2 = x^2 + y^2 = (-1)^2 + (1)^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\tan(\theta) = \frac{1}{-1} = -1 \Rightarrow \theta = \tan^{-1}(-1) = 3\pi/4$$

In polar coordinates the point is $P(1, 3\pi/4)$



Polar Curves

Example 3

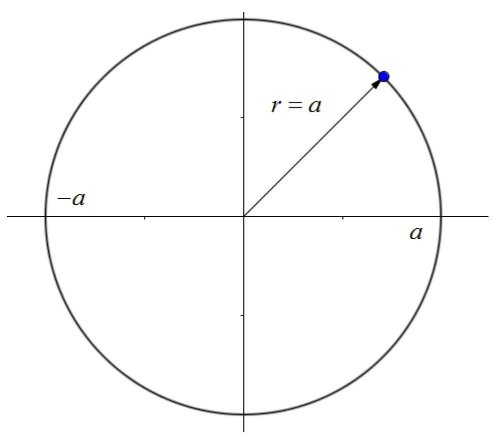
Identify the curve by finding the Cartesian equation. Then sketch the curve.

1. $r = a$
2. $\theta = \pi/4$
3. $r = 4\sin(\theta)$
4. $r = 2(1 + \cos(\theta))$

1. $r = a$

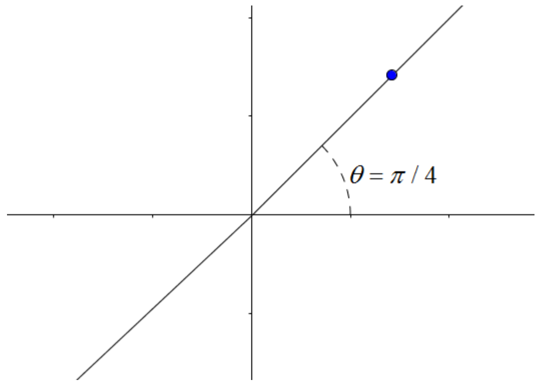
   Since this formula does not rely on θ it means that every point on this curve (no matter θ) is $r = a$. Every point has the form (a, θ). Basically it means it's a circle with radius a.

   $$r^2 = x^2 + y^2 = a^2$$



2. $\theta = \pi/4$

   Since this formula doesn't rely on r it means that every point on this curve has the form $(r, \pi/4)$. For example $(1, \pi/4)$, $(2, \pi/4)$, and $(-1, \pi/4)$.

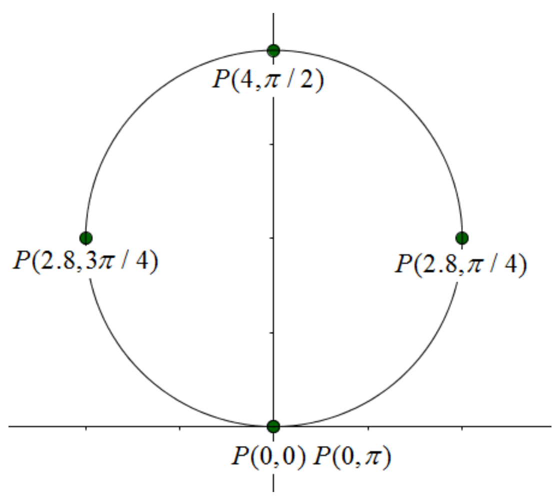


3. $r = 4\sin(\theta)$

Functions like this can be handled two ways. The less complicated ones we can simplify make a t table with r and θ to generate a few points to plot.

θ	r	$P(r, \theta)$
0	$4\sin(0) = 0$	$P(0, 0)$
$pi/4$	$4\sin(\pi/4) = 2\sqrt{2} \approx 2.8$	$P(2.8, \pi/4)$
$\pi/2$	$4\sin(\pi/2) = 4$	$P(4, \pi/2)$
$3\pi/4$	$4\sin(3\pi/4) = 2\sqrt{2} \approx 2.8$	$P(2.8, 3\pi/4)$
π	$4\sin(\pi) = 0$	$P(0, \pi)$
$5\pi/4$	$4\sin(5\pi 4) = -2\sqrt{2} \approx -2.8$	$P(-2.8, 5\pi/4)$

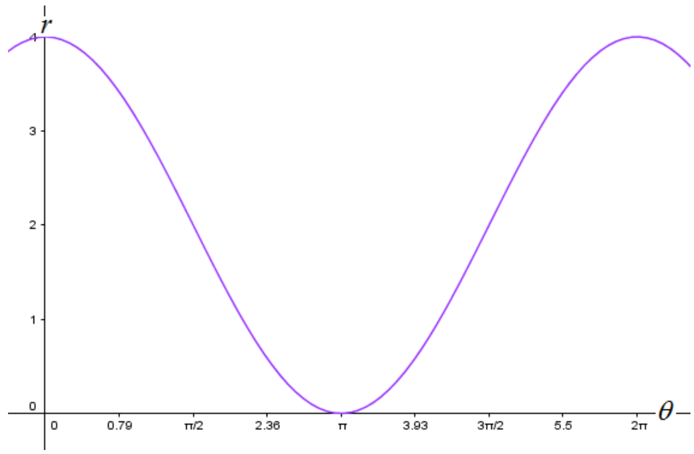
Plotting these points and connecting the dots we get



Notice that the curve does a full revolution after $\theta = \pi$. Remember that polar coordinates are not unique. For example $P(-2.8, 5\pi/4) = P(2.8, \pi/4)$.

4. $r = 2(1 + \cos(\theta))$.

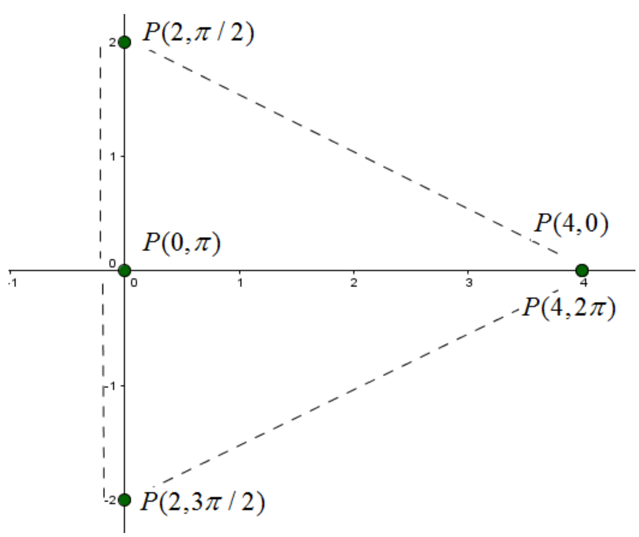
Plotting some points can be helpful but it may not give you all the information you need. We need a visual relationship between r and θ. Consider the graph of $r = 2(1 + \cos(\theta))$ on the $r\theta$ plane. THIS IS NOT THE xy plane that we eventually want.



Let's look for all the important points. These include any points that were important from calculus 1. For example intercepts, maximums, and minimums. The easy points to see are

$$P(0,4),\ P(2,\pi/2),\ P(0,\pi),\ P(2,3\pi/2),\ P(4,2\pi)$$

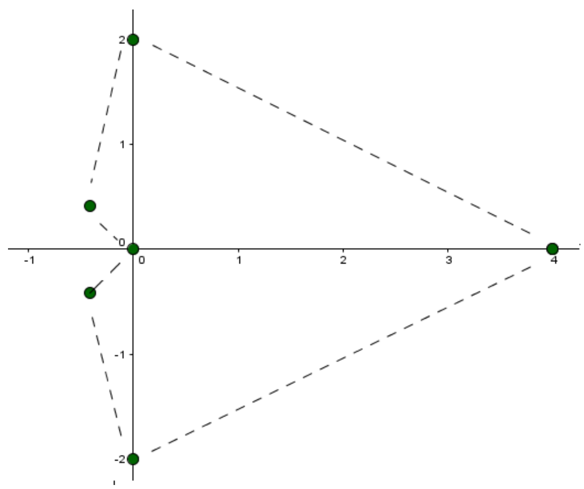
If you try plotting these you can something that looks like

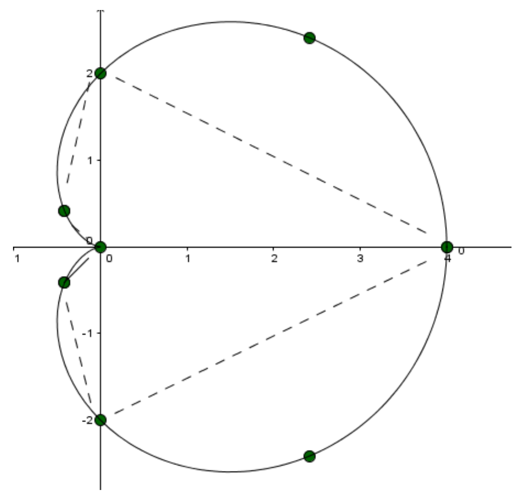


It appears something should be happening in the 2nd and 3rd quadrant. The points we choose don't show us. The 2nd quadrant part of the graph occurs when $\pi/2 \le \theta \le \pi$.

The graph of $r = 2(1 + \cos(\theta))$ on the $r\theta$ plane shows us that between $[\pi/2, \pi]$ that the radius r is positive. That means as θ points us towards the 2nd quadrant a positive radius keeps us in the 2nd quadrant. Let's go ahead and plot the point where $\theta = 3\pi/4$. Also as $\theta = 5\pi/4$ points us towards the 3rd quadrant we know the radius r is also positive. This should give us a point in the 3rd quadrant.

I added the points in the left graph below to get a better idea. In the right graph I connected the dots with a smooth curve. I also added the points when $\theta = \pi/4$ and $\theta = 7\pi/4$.



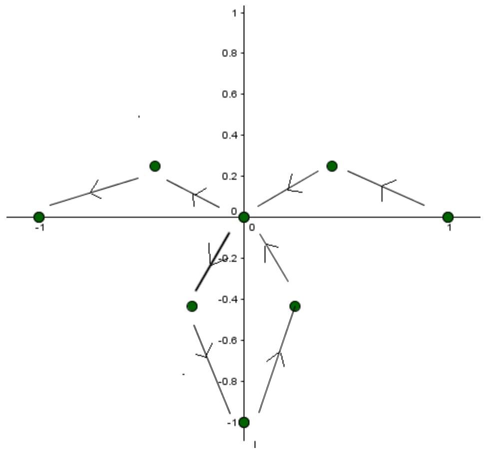


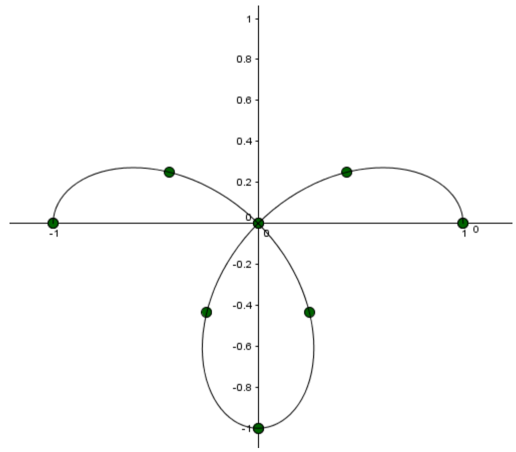
I guess if you want to take anything away from this is **PLOT LOTS OF POINTS**. I mean a lot.

Example 4

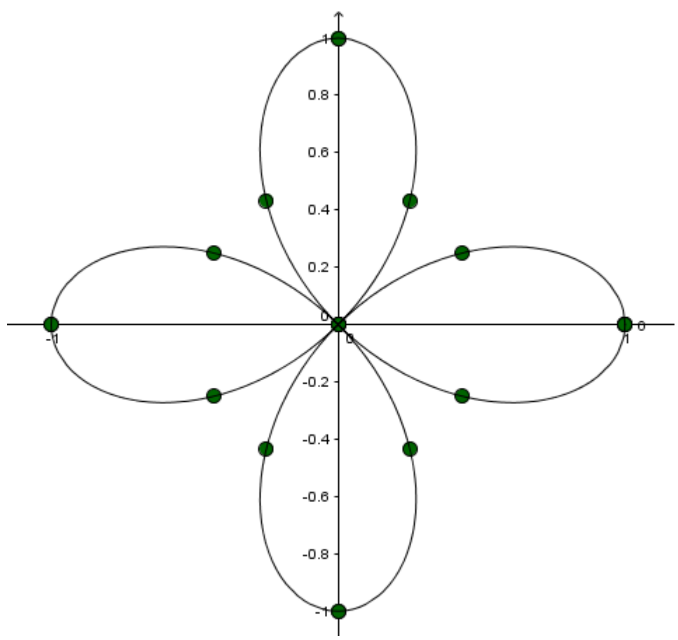
Sketch $r = \cos(2\theta)$.

To do this graph I started by plotting points for $\theta = 0, \pi/6, \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4, 5\pi/6, \pi$. Make sure you are connecting the dots as you pHere's what I got:





It looks like the graph is only half way done. AND I PLOTTED 9 points. And two of them actually was the origin. Could you imagine how inaccurate the group would look if I used fewer points. So then I plotted points for $\theta = 7\pi/6, 5\pi/4, 4\pi/4, 3\pi/2, 5\pi/3, 7\pi/4, 2\pi$. Plotting the additional points and connecting the dots I get



Tangents To Polar Curves

Definition 1: Derivative of Polar Curves

Let $r = f(\theta)$. Recall that

$$x = r\cos(\theta) = f(\theta)\cos(\theta)$$

$$y = r\sin(\theta) = f(\theta)\sin(\theta)$$

then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Note that

$$\frac{dy}{d\theta} = f'(\theta)\sin(\theta) + f(\theta)\cos(\theta) = \frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)$$
$$\frac{dx}{d\theta} = f'(\theta)\cos(\theta) - f(\theta)\sin(\theta) = \frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)$$

Putting this all together we get

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

Definition 2: Horizontal and Vertical Tangents

$r = f(\theta)$ has a **Horizontal Tangent** when $\frac{dy}{d\theta} = 0$, provided $\frac{dx}{d\theta} \neq 0$.

$r = f(\theta)$ has a **Vertical Tangent** when $\frac{dx}{d\theta} = 0$, provided $\frac{dy}{d\theta} \neq 0$.

Example 5

Find the slope of the tangent line of $r = 2 + \sin(3\theta)$ at $\pi/4$.

1. First thing, let's find write our the formula

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

$$\frac{dy}{dx} = \frac{3\cos(3\theta)\sin(\theta) + (2+\sin(3\theta))\cos(\theta)}{3\cos(3\theta)\cos(\theta) - (2+\sin(\theta))\sin(\theta)}$$

You can try to simplify if you want. I think I'll leave it.

2. Next, plug in $\theta = \pi/4$.

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{3\cos(3\pi/4)\sin(\pi/4) + (2+\sin(3\pi/4))\cos(\pi/4)}{3\cos(3\pi/4)\cos(\pi/4) - (2+\sin(\pi/4))\sin(\pi/4)}$$

$$= \frac{3\cdot -\frac{1}{2} + \left(2+\frac{\sqrt{2}}{2}\right)\cdot\frac{\sqrt{2}}{2}}{3\cdot -\frac{1}{2} - \left(2+\frac{\sqrt{2}}{2}\right)\cdot\frac{\sqrt{2}}{2}}$$

$$= \frac{\sqrt{2}-1}{-\sqrt{2}-1}$$

Example 6

Let $r = 3\cos(\theta)$. Find the slope of the tangent at $\theta = \pi/6$. Then find the horizontal and vertical tangents.

$$\begin{aligned}\frac{dy}{dx} &= \frac{-3\sin(\theta)\sin(\theta) + 3\cos(\theta)\cos(\theta)}{-3\sin(\theta)\cos(\theta) - 3\cos(\theta)\sin(\theta)} \\ &= \frac{3\cos^2(\theta) - 3\sin^2(\theta)}{-6\sin(\theta)\cos(\theta)} \\ &= \frac{\cos^2(\theta) - \sin^2(\theta)}{-2\sin(\theta)\cos(\theta)}\end{aligned}$$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{\theta=\pi/6} &= \frac{\cos^2(\pi/6) - \sin^2(\pi/6)}{-2\sin(\pi/6)\cos(\pi/6)} \\ &= \frac{3/4 - 1/4}{-2\cdot 1/2\cdot\frac{\sqrt{3}}{2}} \\ &= -\frac{1}{\sqrt{3}}\end{aligned}$$

Now to find the horizontal and vertical tangents.

$$\frac{dy}{d\theta} = \cos^2(\theta) - \sin^2(\theta) = 0?$$

$$\begin{aligned} \cos^2(\theta) - \sin^2(\theta) &= 0 \\ \cos^2(\theta) &= \sin^2(\theta) \\ 1 &= \tan^2(\theta) \\ \pm 1 &= \tan(\theta) \end{aligned}$$

$$\theta = \pi/4, 3\pi/4$$

Horizontal Tangents in Polar coordinates: $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{4}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{3\pi}{4}\right)$.

Horizontal Tangents in Rectangular coordinates: $(3/2, 3/2)$ and $(3/2, -1.5)$

Technically θ can also be $\theta = 5\pi/4, 7\pi/4$ but you'll see from the graph that it repeats itself after $\theta = \pi$.

To find the vertical tangents we need to solve

$$\frac{dx}{d\theta} = -2\sin(\theta)\cos(\theta) = -\sin(2\theta) = 0$$

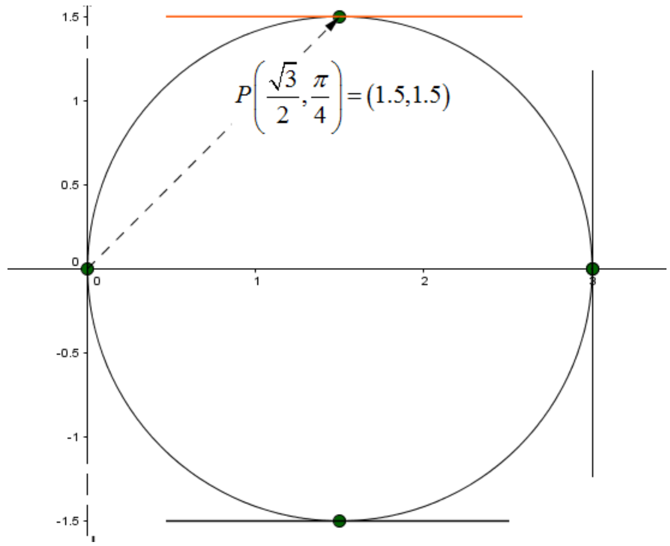
This occurs when $2\theta = 0, \pi, 2\pi$,etc. Solving for θ we get

$$\theta = 0, \pi/2, \pi$$

. Vertical Tangents in Polar coordinates: $(3, 0)$, $(0, \pi/2)$, $(-3, \pi)$. Note that $P(3, 0) = P(-3, \pi)$.

Vertical Tangents in Rectangular coordinates: $(0, 0)$, $(3, 0)$

Here's the graph:


$P\left(\frac{\sqrt{3}}{2}, \frac{\pi}{4}\right) = (1.5, 1.5)$
1.5
1
0.5
0
-0.5
-1
-1.5
0
1
2
3