Math 232

Calculus III

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10.2 Calculus with Parametric Curves

Definition 1: First Derivative of a Parametric Curve

Suppose f and g are differentiable functions where $x = f(t)$ and $y = g(t)$. Then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \text{ The Chain Rule}$$

Rearranging the terms we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ provided } \frac{dx}{dt} \neq 0$$

To find the second derivative:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \end{aligned}$$

Rearrange the top part of the derivative and we get

Definition 2: Second Derivative of a Parametric Curve

Suppose f and g are differentiable functions where $x = f(t)$ and $y = g(t)$. Then the second derivative is

$$\frac{dy}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example 1

Let $x = t^5 - 4t^3$ and $y = t^2$

1. Find $\dfrac{dy}{dx}$

2. Find the tangent line(s) at $(0, 4)$

1. Find $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ since these are parts of the formula for $\dfrac{dy}{dx}$

$$\frac{dy}{dt} = 2t$$

$$\frac{dx}{dt} = 5t^4 - 12t^2$$

Putting these together

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

2. Note the derivative is in terms of t. To use the derivative we need to know what value(s) of t will give us the point (0,4). We need t values that satisfy both $x = 0$ and $y = 4$

$$t^2 = 4 \Rightarrow t = -2, 2$$

$$t^5 - 4t^3 = t^3(t^2 - 4) = 0 \Rightarrow t = 0, -2, 2$$

The only values of t that work for both are $t = -2, 2$. This means we cross the point $(0, 4)$ twice. That also means there are two tangent lines.

(a) Tangent Line 1: We need a slope and a point

$$\left.\frac{dy}{dx}\right|_{t=-2} = \frac{2}{5(-2)^3 - 12(-2)} = -\frac{1}{8}$$

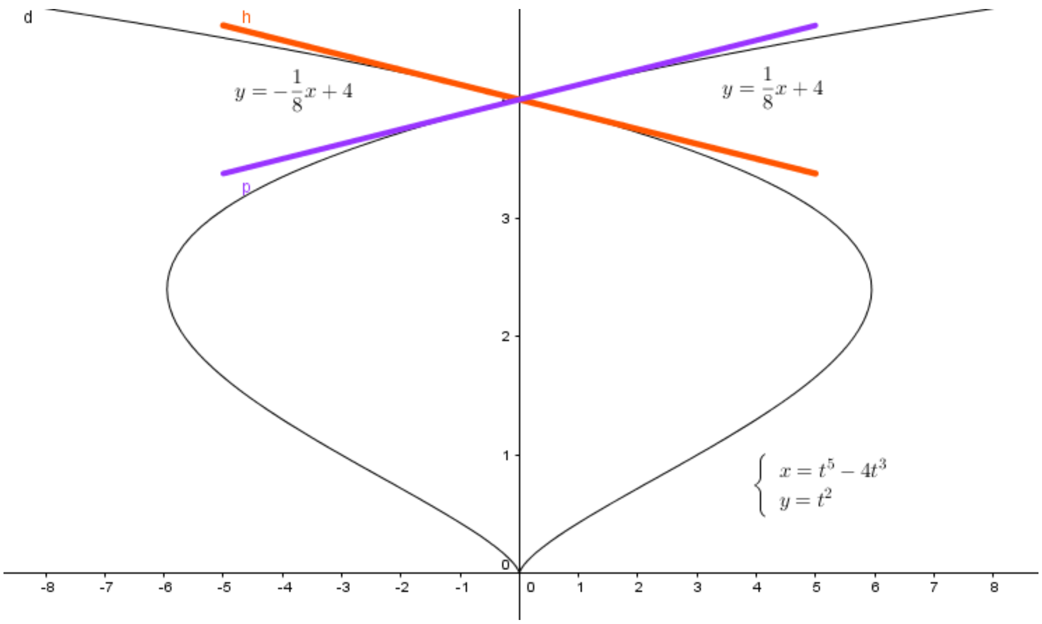
Tangent Line: $y - 4 = -\dfrac{1}{8}(x - 0) \Rightarrow y = -\dfrac{1}{8}x + 4$

(b) Tangent Line 2: We need a slope and a point

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{2}{5(2)^3 - 12(2)} = \frac{1}{8}$$

$$\text{Tangent Line: } y - 4 = \frac{1}{8}(x - 0) \Rightarrow y = \frac{1}{8}x + 4$$

Let's look at the graph to verify this:



Definition 3: Horizontal and Veritical Tangents

$x = x(t)$ and $y = y(t)$ has a **Horizontal Tangent Line** when

$$\frac{dy}{dt} = 0 \text{ provided } \frac{dx}{dt} \neq 0$$

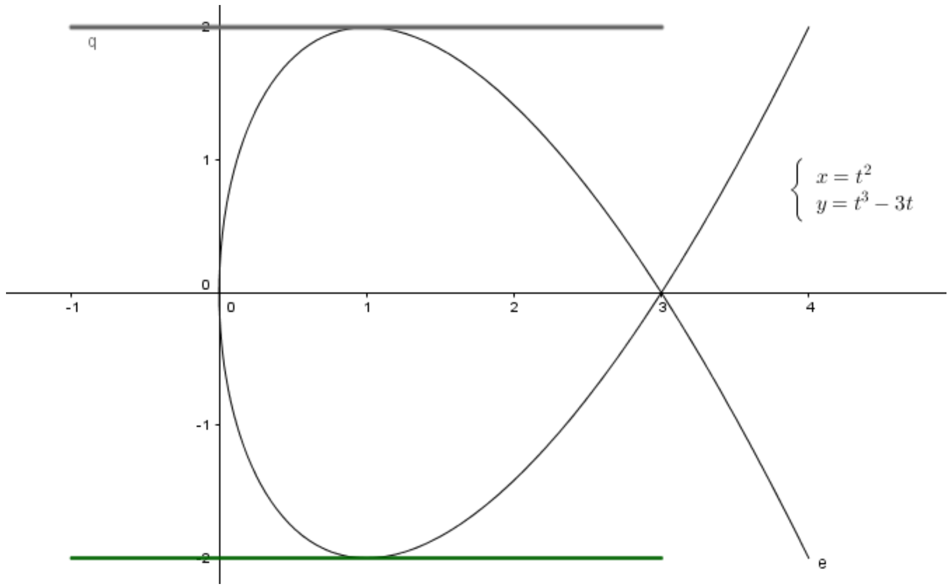
And a **Vertical Tangent Line** when

$$\frac{dx}{dt} = 0 \text{ provided } \frac{dy}{dt} \neq 0$$

Example 2

Let C be a curve defined by $x = t^2$ and $y = t^3 - 3t$. Find the points where a horizontal or vertical tangent line occurs.

Let's look the graph before getting to the math.



It appears there are two horizontal tangents at (1,2) and (1,-2). There is one vertical tangent at (0,0). Let's show this using the derivative.

1. Horizontal Tangents:

$$\frac{dy}{dt} = 3t^2 - 3 = 3(t^2 - 1) = 3(t-1)(t+1)$$

We have two horizontal tangents at $t = -1$ and $t = 1$. As points that would be $(x(-1), y(-1)) = (1, 2)$ and $(x(1), y(1)) = (1, -2)$.

2. Vertical Tangents:

$$\frac{dx}{dt} = 2t = 0$$

We have a vertical tangent when $t = 0$. As a point that is $(x(0), y(0) = (0, 0)$.

Example 3: Using the previous example

1. Find the second derivative $\frac{d^2y}{dx^2}$.
2. Determine the values of t for which the curve is concave up and concave down.

1. Recall that $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

$$\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right) = \frac{2t(6t) - (3t^2-3)(2)}{4t^2}$$

$$= \frac{6t^2+6}{4t^2}$$
$$= \frac{3}{2} + \frac{3}{2t^2}$$

$$\text{And } \frac{dx}{dt} = 2t$$

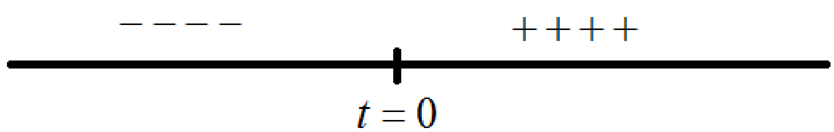
$$\text{So } \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{d^2y}{dx^2} = \frac{\frac{3}{2}+\frac{3}{2t^2}}{2t} = \frac{3}{4t} + \frac{3}{4t^3} = \frac{3t^2+3}{4t^3}$$

2. To determine the values of t where the curve is concave down we need to find the critical values for $\frac{d^2y}{dx^2}$.

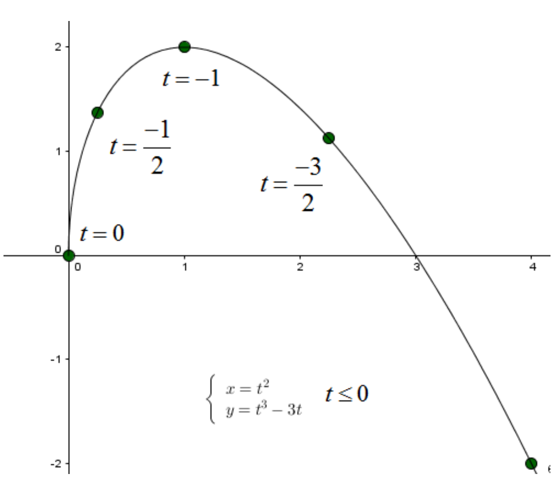
$$3t^2 + 3 = 0 \text{ has no real solutions}$$

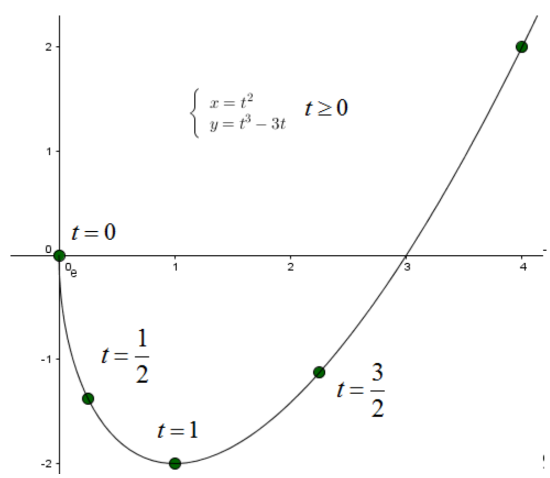
$$\frac{3t^2+3}{4t^3} \text{ DNE when } t = 0$$

Let's set up a number line for t



So the curve C is concave up on $t \in (0, \infty)$ and concave down when $t \in (-\infty, 0)$. Below you'll find the two graphs on the two domains.





Example 4

Let $x = 1 + \ln(t)$ and $y = t^2 + 2$.

1. Find the tangent line at $(1, 3)$ by NOT eliminating t.
2. Find the tangent line at $(1, 3)$ by first eliminating t.

1. Let's find $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\frac{1}{t}} = 2t^2$$

What value of t gives us the point $(1, 3)$? $t = 1$

The slope when $t = 1$ is $\left.\dfrac{dy}{dx}\right|_{t=1} = 2(1)^2 = 2$

The tangent line is $y - 3 = 2(x - 1) \Rightarrow y = 2x + 1$

2. To eliminate t we should try to solve for t using either $x = 1 + \ln(t)$ or $y = t^2 + 2$. Because of the square root in y, I think I'll solve for t using $x = 1 + \ln(t)$.

$$\begin{aligned} x &= 1 + \ln(t) \\ x - 1 &= \ln(t) \\ e^{x-1} &= e^{\ln(t)} \\ e^{x-1} &= t \end{aligned}$$

Next, plug $t = e^{x-1}$ into $y = t^2 + 2$ to get $y = (e^{x-1})^2 + 2$. We have now eliminated t. Now let's find $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = 2e^{x-1} \cdot 1 + 0 = 2e^{x-1}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 2e^{1-1} = 2e^0 = 2$$

The tangent line is $y - 3 = 2(x - 1) \Rightarrow y = 2x + 1$

Arc Length

Formula 1: Arc Length

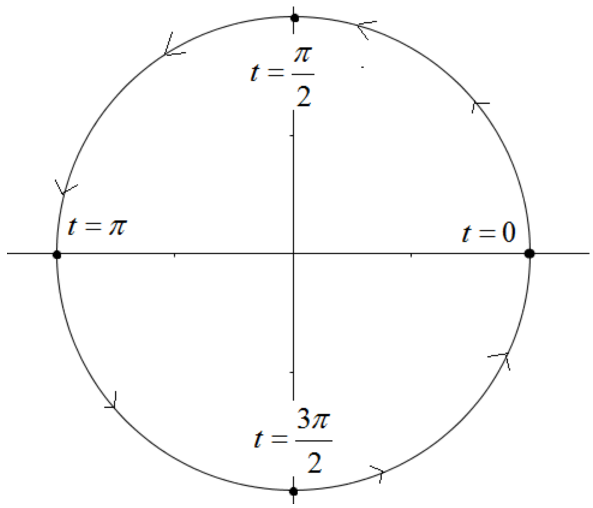
If C is described by $x = f(t)$ and $y = g(t)$ on $\alpha \leq t \leq \beta$ and are continuous and C is traversed exactly once as t increases, then

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 5

Let $x = \cos(t)$ and $y = \sin(t)$, $0 \leq t \leq 2\pi$. Find the arc length.

Note that this curve is a circle centered at $(0, 0)$ with radius 1.



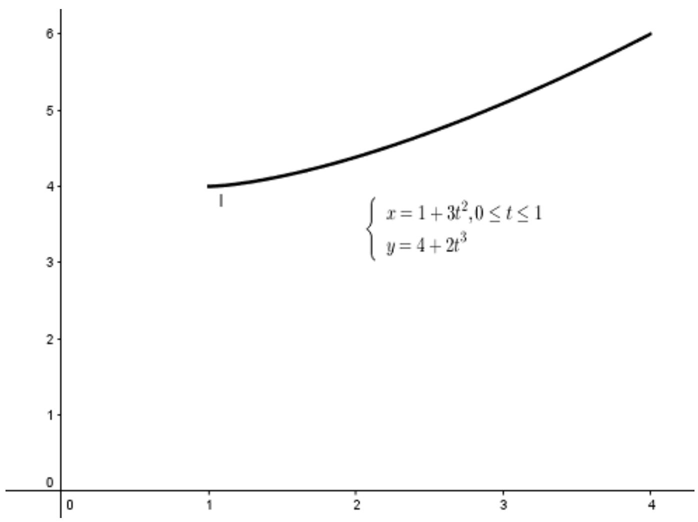
We know the arc length (circumference) is $2\pi(1) = 2pi$. Let's see if the formula gives us the same thing.

$$\frac{dx}{dt} = -\sin(t)$$
$$\frac{dy}{dt} = \cos(t)$$

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t)}\, dt \\ &= \int_0^{2\pi} \sqrt{1}\, dt \\ &= \int_0^{2\pi} 1\, dt \\ &= t\Big|_0^{2\pi} \\ &= 2\pi \end{aligned}$$

Example 6

Find the arc length of $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \le t \le 1$.



$$\frac{dy}{dt} = 6t$$
$$\frac{dx}{dt} = 6t^2$$

$$\begin{aligned} L &= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt \\ &= \int_0^1 \sqrt{36t^2 + 36t^4}\, dt \\ &= \int_0^1 \sqrt{36t^2(1+t^2)}\, dt \\ &= \int_0^1 6t\sqrt{1+t^2}\, dt \end{aligned}$$

This is a perfect candidate for u-substitution.

1. Let $u = 1 + t^2$
2. $du = 2t\, dt \Rightarrow \frac{1}{2}\, du = t\, dt$
3. If $t = 1$, $u = 2$
4. If $t = 0$, $u = 1$

$$\begin{aligned} L &= \int_0^1 6t\sqrt{1+t^2}\, dt \\ &= \int_1^2 3u^{1/2}\, du \\ &= 2u^{3/2}\Big|_1^2 \\ &= 2(2)^{3/2} - 2(1)^{3/2} \\ &= 2(2^{3/2} - 1) \\ &\approx 3.66 \end{aligned}$$